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A DIVISOR PROBLEM

By F. V. ATKINSON (*Oxford*)

[Received 17 June 1941]

1. Let us denote by $d_k(n)$ the number of ways of expressing n as the product of k factors. Then we have the asymptotic formula*

$$\sum_{n<x} d_k(n) = xf_k(\log x)+\Delta_k(x), \tag{1.1}$$

where $f_k(\log x)$ is a polynomial of degree $(k-1)$ in $\log x$, the residue of $\zeta^k(s)\,x^{s-1}/s$ at the pole $s = 1$, and

$$\Delta_k(x) = O(x^\alpha), \tag{1.2}$$

for some $\alpha < 1$. Let us denote by α_k the lower bound of numbers α such that (1.2) is true, for any given k. Then it is known† that

$$\alpha_k \leqslant \max\left(\frac{1}{2}, \frac{k-1}{k+2}\right). \tag{1.3}$$

For $k \geqslant 4$ this is a better result than the earlier one of Landau,‡

$$\alpha_k \leqslant \frac{k-1}{k+1}; \tag{1.4}$$

the results are the same for $k = 3$, viz. $\alpha_3 \leqslant \frac{1}{2}$, and Landau's is the better one for $k = 2$, but in each case Landau's method allows of improvement by the use of van der Corput's method. The improvement is, however, very small and, for larger values of k, Hardy and Littlewood's result is still the best. We prove here the following

Theorem. $\alpha_3 \leqslant \frac{37}{75}$.

This is slightly better than the result $\alpha_3 \leqslant \frac{43}{87}$, obtained by Walfisz§ by the Weyl-Hardy-Littlewood method.

2. Since $d_3(n) = O(n^\epsilon)$ for any fixed positive ϵ, we may take x to be half an odd integer, without loss of generality. Then we have, by Perron's formula,

$$\Delta_3(x)+\frac{1}{8} = \frac{1}{2\pi i}\int\limits_C \zeta^3(s)\frac{x^s}{s}\,ds, \tag{2.1}$$

* Piltz (**4**).

† G. H. Hardy and J. E. Littlewood (**2**).

‡ E. Landau (**3**).

§ A. Walfisz (**7**).

where C is the contour formed by the five lines joining the points $1+\delta-i\infty$, $1+\delta-ix^{\alpha}$, $-\delta-ix^{\alpha}$, $-\delta+ix^{\alpha}$, $1+\delta+ix^{\alpha}$, $1+\delta+i\infty$, and δ, α are fixed positive numbers to be chosen later, with $\frac{1}{2}<\alpha<\frac{2}{3}$. Let the respective parts of the contour be denoted by C_1, C_2, C_3, C_4, C_5.

Now
$$\int_{C_5} \zeta^3(s)\frac{x^s}{s}\,ds = \sum_{n=1}^{\infty} d_3(n) \int_{1+\delta+ix^{\alpha}}^{1+\delta+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s}.$$

But
$$\int_{1+\delta+ix^{\alpha}}^{1+\delta+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \int_{(1+\delta+ix^{\alpha})\log(x/n)}^{(1+\delta+i\infty)\log(x/n)} e^s\frac{ds}{s} = O[x^{1+\delta-\alpha}n^{-1-\delta}\{\log(x/n)\}^{-1}].$$

Hence
$$\int_{C_5} \zeta^3(s)\frac{x^s}{s}\,ds = O\Big(x^{1+\delta-\alpha}\sum_{n=1}^{\infty} d_3(n)n^{-1-\delta}|\log(x/n)|^{-1}\Big).$$

In this infinite sum we put
$$\sum_{n=1}^{\infty} = \sum_{n<\frac{1}{2}x} + \sum_{\frac{1}{2}x<n<x} + \sum_{x<n\leqslant 2x} + \sum_{n>2x} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4.$$

In $\sum_1$, $\{\log(x/n)\}^{-1} = O(1)$, and hence
$$\sum\nolimits_1 = O\Big(\sum_{n<\frac{1}{2}x} d_3(n)n^{-1-\delta}\Big) = O(1),$$
and similarly for $\sum_4$.

In $\sum_2$, $\left(\log\dfrac{x}{n}\right)^{-1} = O\left(\dfrac{x}{x-n}\right)$, $d_3(n) = O(x^{\delta})$, $(n^{-1-\delta}) = O(x^{-1-\delta})$, and hence
$$\sum\nolimits_2 = O\Big(\sum_{\frac{1}{2}x<n<x} \frac{1}{x-n}\Big) = O(\log x),$$
since x is half an odd integer. Similarly for $\sum_3$. Hence
$$\int_{C_5} \zeta^3(s)\frac{x^s}{s}\,ds = O(x^{1+\delta-\alpha}\log x), \tag{2.2}$$
and similarly for C_1.

Now
$$\int_{C_4} \zeta^3(s)\frac{x^s}{s}\,ds = \left(\int_{-\delta+ix^{\alpha}}^{ix^{\alpha}} + \int_{ix^{\alpha}}^{1+ix^{\alpha}} + \int_{1+ix^{\alpha}}^{1+\delta+ix^{\alpha}}\right)\zeta^3(s)\frac{x^s}{s}\,ds.$$

We apply here the following well-known order-results,

$$\zeta(\sigma+it) = \begin{cases} O(t^{\frac12-\sigma}\log t) & (\sigma \leqslant 0), & (2.3)\\ O(t^{\frac12-\frac12\sigma}\log t) & (0 \leqslant \sigma \leqslant 1), & (2.4)\\ O(\log t) & (\sigma \geqslant 1), & (2.5)\end{cases}$$

these results holding uniformly in the given ranges of σ. Hence, by (2.3),

$$\int\limits_{-\delta+ix^\alpha}^{ix^\alpha} \zeta^3(s)\frac{x^s}{s}\,ds = O\left(\int\limits_0^\delta x^{(\frac32+3\sigma)\alpha-\sigma-\alpha}\log^3 x\,d\sigma\right)$$

$$= O(x^{\frac12\alpha+\delta(3\alpha-1)}\log^2 x)$$

$$= O(x^{\frac12+\delta}), \quad \text{since } \alpha < \tfrac23.$$

Next, by (2.4),

$$\int\limits_{ix^\alpha}^{1+ix^\alpha} \zeta^3(s)\frac{x^s}{s}\,ds = O\left(\int\limits_0^1 x^{\frac32\alpha(1-\sigma)+\sigma-\alpha}\log^3 x\,d\sigma\right) = O(x^{1-\alpha}\log^3 x).$$

Lastly, by (2.5),

$$\int\limits_{1+ix^\alpha}^{1+\delta+ix^\alpha} \zeta^3(s)\frac{x^s}{s}\,ds = O\left(\int\limits_1^{1+\delta} x^{\sigma-\alpha}\log^3 x\,d\sigma\right) = O(x^{1+\delta-\alpha}\log^2 x).$$

Hence
$$\int\limits_{C_4} \zeta^3(s)\frac{x^s}{s}\,ds = O(x^{1+\delta-\alpha}\log^2 x), \tag{2.6}$$

and similarly for C_2.

Thus we have, by (2.1), (2.2), (2.6),

$$\Delta_3(x) = \frac{1}{2\pi i}\int\limits_{-\delta-ix^\alpha}^{-\delta+ix^\alpha} \zeta^3(s)\frac{x^s}{s}\,ds + O(x^{1+\delta-\alpha}\log^2 x)$$

$$= \frac{1}{2\pi i}8x\int\limits_{1+\delta-ix^\alpha}^{1+\delta+ix^\alpha} \Gamma^3(s)\cos^3\tfrac12 s\pi(8\pi^3x)^{-s}\zeta^3(s)\frac{ds}{1-s} + O(x^{1+\delta-\alpha}\log^2 x). \tag{2.7}$$

3. We have, by Stirling's formula,

$$\log\frac{\Gamma^3(s)}{s-1} = (3s-\tfrac52)\log s - 3s + \tfrac32\log 2\pi + O(s^{-1})$$

$$= (3s-\tfrac52)\log(3s-2) - (3s-2) + \tfrac32\log 2\pi - (3s-\tfrac52)\log 3 + O(s^{-1})$$

$$= \log\Gamma(3s-2) + \log 2\pi - (3s-\tfrac52)\log 3 + O(s^{-1}).$$

Also
$$8\cos^2\tfrac12 s\pi = 2\cos\tfrac32 s\pi\{1+O(e^{-\frac12|s|\pi})\}$$

$$= -2\cos\tfrac12(3s-2)\pi\{1+O(e^{-\frac12|s|\pi})\}.$$

Hence

$$8\frac{\Gamma^3(s)}{1-s}\cos^3\tfrac{1}{2}s\pi = 4\pi\sqrt{3}\Gamma(3s-2)3^{2-3s}\cos\{\tfrac{1}{2}(3s-2)\pi\}\{1+O(s^{-1})\}. \tag{3.1}$$

Now, if $s = 1+\delta\pm it$, we have, again by Stirling's formula,

$$\frac{\Gamma^3(s)}{1-s}\cos^3\tfrac{1}{2}s\pi = O(t^{3\delta+\frac{1}{2}}),$$

and so, if we substitute (3.1) in (2.7), the error term in (3.1) will give us

$$O\left(x^{-\delta}\int\limits_0^{x^\alpha} t^{3\delta-\frac{1}{2}}\,dt\right) = O(x^{\frac{1}{2}\alpha+\delta(3\alpha-1)}) = O(x^{1+\delta-\alpha}).$$

Hence, by (2.7) and (3.1),

$$\Delta_3(x) = \frac{4\pi x\sqrt{3}}{2\pi i}\int\limits_{1+\delta-ix^\alpha}^{1+\delta+ix^\alpha}\Gamma(3s-2)\cos\{\tfrac{1}{2}(3s-2)\pi\}3^{2-3s}(8\pi^3x)^{-s}\zeta^3(s)\,ds+ \\ +O(x^{1+\delta-\alpha}\log^2x).$$

Changing the variable to $\frac{1}{3}s+\frac{2}{3}$ and using the Dirichlet series for $\zeta^3(s)$, we have

$$\Delta_3(x) = \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}}\sum_{n=1}^\infty\frac{d_3(n)}{n^{\frac{2}{3}}}\frac{1}{2\pi i}\int\limits_{1+3\delta-3ix^\alpha}^{1+3\delta+3ix^\alpha}\Gamma(s)\cos(\tfrac{1}{2}s\pi)\{6\pi(nx)^{\frac{1}{3}}\}^{-s}\,ds+ \\ +O(x^{1+\delta-\alpha}\log^2x)$$

$$= \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}}\sum_{n=1}^\infty\frac{d_3(n)}{n^{\frac{2}{3}}}I_n+O(x^{1+\delta-\alpha}\log^2x). \tag{3.2}$$

4. I shall now prove two lemmas. Let us denote $\frac{1}{8}x^{3\alpha-1}/\pi^3$ by X.

LEMMA 1. *For $n \leqslant X$,*

$$I_n = \cos 6\pi(nx)^{\frac{1}{3}}+O\left\{(nx)^{-\frac{1}{3}}\left(\log\frac{X}{n}\right)^{-1}\right\}+O(x^{\frac{1}{2}\alpha-\frac{1}{3}+\delta}n^{-\frac{1}{3}-\delta}),$$

and also

$$= \cos 6\pi(nx)^{\frac{1}{3}}+O(x^{\frac{1}{2}\alpha-\frac{1}{3}}n^{-\frac{1}{3}}).$$

We have

$$I_n = \frac{1}{2\pi i}\left\{\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}-\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}-3ix^\alpha}-\int\limits_{\frac{1}{2}-3ix^\alpha}^{1+3\delta-3ix^\alpha}-\int\limits_{1+3\delta+3ix^\alpha}^{\frac{1}{2}+3ix^\alpha}-\int\limits_{\frac{1}{2}+3ix^\alpha}^{\frac{1}{2}+i\infty}\right\}\Gamma(s)\times \\ \times\cos(\tfrac{1}{2}s\pi)\{6\pi(nx)^{\frac{1}{3}}\}^{-s}\,ds$$

$$= A_1-A_2-A_3-A_4-A_5,\quad\text{say.} \tag{4.1}$$

Now $A_1 = \cos 6\pi(nx)^{\frac{1}{2}}$. To deal with the other terms we use Stirling's formula. This gives

$$\Gamma(\sigma+it)$$
$$= t^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi t}\sqrt{(2\pi)}\exp i\left\{t\log t-t+\tfrac{1}{2}\pi(\sigma-\tfrac{1}{2})+\frac{\sigma-\sigma^2-\frac{1}{6}}{2t}+O(t^{-2})\right\}.$$

Therefore

$$A_5 = (nx)^{-\frac{1}{2}}\int\limits_{3x^\alpha}^{\infty}\exp i\left\{t\log t-t-t\log\{6\pi(nx)^{\frac{1}{2}}\}+\frac{1}{24t}\right\}\{1+O(t^{-2})\}\,dt$$

multiplied by a constant. Now the term $O(t^{-2})$ contributes

$$O\left((nx)^{-\frac{1}{2}}\int\limits_{3x^\alpha}^{\infty}t^{-2}\,dt\right) = O(x^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}).$$

If we write $f(t)$ for the function inside the brackets $\{\,\}$, we have

$$f'(t) = \log\frac{t}{6\pi(nx)^{\frac{1}{2}}}-\frac{1}{24t^2} > c_1\log\frac{x^{\alpha-\frac{1}{2}}}{2\pi n^{\frac{1}{2}}} = \tfrac{1}{3}c_1\log\frac{X}{n},$$

for $t > 3x^\alpha$, $n \leqslant X-1$, and some fixed $c_1 > 0$. Also

$$f''(t) = \frac{1}{t}+\frac{1}{12t^3} > \frac{1}{6x^\alpha},$$

for $t < 6x^\alpha$.

It follows from some well-known theorems on integrals* that

$$\int\limits_{3x^\alpha}^{\infty}\exp i\{f(t)\}\,dt = O\left\{\left(\log\frac{X}{n}\right)^{-1}\right\}$$

and also $\qquad = O(x^{\frac{1}{2}\alpha}).$

Therefore

$$A_5 = O\left\{(nx)^{-\frac{1}{2}}\left(\log\frac{X}{n}\right)^{-1}\right\}+O(x^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}), \tag{4.2}$$

and also $\qquad = O(x^{\frac{1}{2}\alpha-\frac{1}{2}}n^{-\frac{1}{2}})+O(x^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}), \qquad (4.3)$

and in each case the second error term may be neglected. Similarly for A_2.

Now $$A_4 = O\left\{\int\limits_{\frac{1}{2}}^{1+3\delta}x^{\alpha(\sigma-\frac{1}{2})}(nx)^{-\frac{1}{2}\sigma}\,d\sigma\right\} = O(x^{\frac{1}{2}\alpha-\frac{1}{4}}n^{-\frac{1}{4}-\delta}), \tag{4.4}$$

and similarly for A_3.

Combining (4.2), (4.3), and (4.4) with (4.1) we get Lemma 1.

* Cf. E. C. Titchmarsh (5), (6).

LEMMA 2. *For $n > X$,*

$$I_n = O\left\{x^{\frac{1}{2}\alpha-\frac{1}{3}+\delta}n^{-\frac{1}{3}-\delta}\left(\log\frac{n}{X}\right)^{-1}\right\},$$

and also $$= O(x^{\alpha-\frac{1}{3}+\delta}n^{-\frac{1}{3}-\delta}).$$

For

$$I_n = \frac{1}{2\pi i}\left\{\int\limits_{1+3\delta-3ix^\alpha}^{1+3\delta-i} + \int\limits_{1+3\delta-i}^{1+3\delta+i} + \int\limits_{1+3\delta+i}^{1+3\delta+3ix^\alpha}\right\}\Gamma(s)\cos(\tfrac{1}{2}s\pi)[6\pi(nx)^{\frac{1}{3}}]^{-s}\,ds$$

$$= B_1+B_2+B_3, \quad \text{say.}$$

Now plainly $$B_2 = O\{(nx)^{-\frac{1}{3}-\delta}\}.$$

The discussion of B_3 is practically the same as that of A_5, except that we have an extra factor $t^{\frac{1}{2}+3\delta}$ in the integrand. Hence we obtain

$$B_3 = O\left\{x^{\frac{1}{2}\alpha-\frac{1}{3}+3\delta(\alpha-\frac{1}{3})}n^{-\frac{1}{3}-\delta}\left(\log\frac{n}{X}\right)^{-1}\right\},$$

and also $$= O(x^{\alpha-\frac{1}{3}+3\delta(\alpha-\frac{1}{3})}n^{-\frac{1}{3}-\delta}).$$

Similarly, we may prove the same inequalities for B_1, and this completes the proof of Lemma 2.

5. Applying Lemmas 1 and 2 in (3.2), we have

$$\begin{aligned}\Delta_3(x) &= \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}}\sum_{n\leqslant X}\frac{d_3(n)}{n^{\frac{2}{3}}}\cos 6\pi(nx)^{\frac{1}{3}}+O(x^{1+\delta-\alpha}\log^2 x)+\\ &\quad +x^{\frac{1}{3}}\sum_{n\leqslant X}\frac{d_3(n)}{n^{\frac{2}{3}}}O\left[\min\left\{\left(\log\frac{X}{n}\right)^{-1}, x^{\frac{1}{2}\alpha}\right\}\right]+\\ &\quad +x^{\frac{1}{2}\alpha+\delta}\sum_{n\leqslant X}d_3(n)\,O(n^{-1-\delta})+\\ &\quad +x^{\frac{1}{2}\alpha+\delta}\sum_{n>X}\frac{d_3(n)}{n^{1+\delta}}O\left[\min\left\{\left(\log\frac{n}{X}\right)^{-1}, x^{\frac{1}{2}\alpha}\right\}\right]\\ &= \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}}\sum_{n<X}\frac{d_3(n)}{n^{\frac{2}{3}}}\cos 6\pi(nx)^{\frac{1}{3}}+O(x^{1+\delta-\alpha}\log^2 x)+\\ &\quad +\sum_{n\leqslant X'-1}\frac{d_3(n)}{n^{\frac{2}{3}}}x^{\frac{1}{3}}O\left\{\left(\log\frac{X}{n}\right)^{-1}\right\}+\\ &\quad +\sum_{n>X'+2}\frac{d_3(n)}{n^{1+\delta}}x^{\frac{1}{2}\alpha+\delta}O\left\{\left(\log\frac{n}{X}\right)^{-1}\right\}+\\ &\quad +O(x^{\frac{1}{2}\alpha+\delta})+O(x^{1-2\alpha+\delta}),\end{aligned}$$

where X' denotes the integral part of X.

Thus

$$\Delta_3(x) = \frac{x^{\frac13}}{\pi\sqrt3}\sum_{n<X}\frac{d_3(n)}{n^{\frac23}}\cos 6\pi(nx)^{\frac13}+O(x^{1+\delta-\alpha}\log^2 x)+\sum\nolimits_1+\sum\nolimits_2,$$

say, remembering that $\frac12 < \alpha < \frac23$.

Put $$\sum\nolimits_1 = \sum_{n\leqslant\frac12 X}+\sum_{\frac12 X<n\leqslant X'-1} = \sum\nolimits_1' + \sum\nolimits_1''.$$

Then, in $\sum_1'$, $\{\log(X/n)\}^{-1} = O(1)$, and so

$$\sum\nolimits_1' = O\Big(x^{\frac13}\sum_{n<x^{3\alpha-1}} d_3(n)\,n^{-\frac12}\Big) = O(x^{\frac12\alpha}\log^2 x).$$

In $\sum_1''$, $\left(\log\frac{X}{n}^{-1}\right) = O\left(\frac{X}{X-n}\right)$, $d_3(n) = O(x^{\delta})$, and so

$$\sum\nolimits_1'' = O\Big(x^{\frac12\alpha+\delta}\sum_{n<X'-1}\frac{1}{X-n}\Big) = O(x^{\frac12\alpha+\delta}\log x).$$

Hence $$\sum\nolimits_1 = O(x^{\frac12\alpha+\delta}\log x) = O(x^{1+\delta-\alpha}\log x),$$

and similarly for $\sum_2$.

Our final approximation to $\Delta_3(x)$ is

$$\Delta_3(x) = \frac{x^{\frac13}}{\pi\sqrt3}\sum_{n<X}\frac{d_3(n)}{n^{\frac23}}\cos 6\pi(nx)^{\frac13}+O(x^{1+\delta-\alpha}\log^2 x), \tag{5.1}$$

where X denotes $x^{3\alpha-1}/8\pi^3$, $\frac12 < \alpha < \frac23$, and $\delta > 0$.

6. Let us write

$$a_n = a_n(x) = \cos 6\pi(nx)^{\frac13}. \tag{6.1}$$

Then

$$\begin{aligned}\sum_{n<x} d_3(n)a_n &= \sum\sum_{pqr<X}\sum a_{pqr}\\ &= 6\sum_{r<X^{\frac13}}\sum_{q<\sqrt{(X/r)}}\sum_{p<X/qr} a_{pqr}-3\sum_{r<X^{\frac13}}\sum_{q<\sqrt{(X/r)}}\sum_{p<\sqrt{(X/r)}} a_{pqr}-\\ &\qquad -3\sum_{r<X^{\frac13}}\sum_{q<X^{\frac13}}\sum_{p<X/qr} a_{pqr}+\sum_{r<X^{\frac13}}\sum_{q<X^{\frac13}}\sum_{p<X^{\frac13}} a_{pqr}\\ &= 6S_1-3S_2-3S_3+3S_4, \quad \text{say.}\end{aligned} \tag{6.2}$$

We now apply Theorem III of Titchmarsh (5). This result gives, with $f(n) = 3(nx)^{\frac13}$, $k = 3$,

$$\sum_{\frac12 a\leqslant n\leqslant a} e^{6\pi i(nx)^{\frac13}} = O(a^{\frac{11}{18}}x^{\frac1{18}}+a^{\frac{17}{18}}x^{-\frac1{18}}).$$

Applying the same result to the ranges $\frac14 a \leqslant n < \frac12 a$; $\frac18 a \leqslant n < \frac14 a$, etc., and adding, we get

$$\sum_{n\leqslant a} e^{6\pi i(nx)^{\frac13}} = O(a^{\frac{11}{18}}x^{\frac1{18}}+a^{\frac{17}{18}}x^{-\frac1{18}}). \tag{6.3}$$

Hence, from (6.2) and (6.3),

$$S_1 = O\left\{\sum_{r<X^{\frac{1}{3}}} \sum_{q<\sqrt{(X/r)}} \left[\left(\frac{X}{qr}\right)^{\frac{31}{42}} (qrx)^{\frac{1}{42}} + \left(\frac{X}{qr}\right)^{\frac{85}{84}} (qrx)^{-\frac{1}{42}}\right]\right\}$$

$$= O\left\{\sum_{r<X^{\frac{1}{3}}} \sum_{q<\sqrt{(X/r)}} \left[X^{\frac{31}{42}} x^{\frac{1}{42}} (qr)^{-\frac{5}{7}} + X^{\frac{85}{84}} x^{-\frac{1}{42}} (qr)^{-\frac{29}{28}}\right]\right\}$$

$$= O\{X^{\frac{13}{14}} x^{\frac{1}{42}} + X^{\frac{85}{84}} x^{-\frac{1}{42}}\},$$

and similarly for the sums S_2, S_3, S_4.

We have then, by (6.2),

$$\sum_{n<X} d_3(n) e^{6\pi i (nx)^{\frac{1}{3}}} = O(X^{\frac{13}{14}} x^{\frac{1}{42}} + X^{\frac{85}{84}} x^{-\frac{1}{42}}).$$

Taking the real part, and using partial summation, we get

$$x^{\frac{1}{3}} \sum_{n<X} \frac{d_3(n)}{n^{\frac{2}{3}}} \cos 6\pi (nx)^{\frac{1}{3}} = O(X^{\frac{11}{42}} x^{\frac{5}{14}} + X^{\frac{29}{84}} x^{\frac{13}{42}}).$$

Hence, by (5.1),

$$\Delta_3(x) = O(x^{\frac{11}{42}(3\alpha-1)+\frac{5}{14}} + x^{\frac{29}{84}(3\alpha-1)+\frac{13}{42}} + x^{1+\delta-\alpha} \log^2 x).$$

Taking $\alpha = \frac{38}{75}$, the first and third terms on the right-hand side become both $O(x^{\frac{37}{75}+\delta} \log^2 x)$, and the second term becomes of lower order than the other two. This gives

$$\Delta_3(x) = O(x^{\frac{37}{75}+\delta} \log^2 x),$$

where δ is as small as we please. This is equivalent to the original theorem.

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HYPERGEOMETRIC SERIES AND 'SET' NUMBERS

By F. H. JACKSON (*Eastbourne*)

[Received 19 September 1941]

1. Introduction

If we have an infinite sequence of elements

$$p_1, p_2, p_3, \ldots, p_r, \ldots,$$

the set number (n) may be used to denote the sum of the elements

$$p_1+p_2+p_3+\ldots+p_n.$$

It is remarkable that when the 'set' number (n) is used to replace the ordinary integer n, in hypergeometric series, many properties of the generalized series are identical with the corresponding properties of the simple hypergeometric series. If, moreover, we form basic 'set' numbers $[(n)]$ such that

$$[(n)] \equiv \frac{q^{(n)}-1}{q-1},$$

we find a correspondence between certain fundamental properties of such basic 'set' series and the well-known basic (Heinean) hypergeometric series. For example, using a notation

$$\mathfrak{F}(\alpha,\beta;\gamma;\lambda,x,p_1,p_2,\ldots) = \sum_{r=0}^{\infty} \frac{[\alpha]_n[\beta]_n}{[(n)]!\,[\gamma]_n} \lambda^n x^{(n)},$$

in which

$$[\alpha]_n = [\alpha][\alpha+(1)][\alpha+(2)][\alpha+(3)]\ldots[\alpha+(n-1)],$$
$$(n) = p_1+p_2+\ldots+p_n,$$

and

$$\mathrm{F}(\alpha,\beta;\gamma;\lambda,x,p_1,p_2,\ldots) = \sum_{r=0}^{\infty} \frac{[\alpha]_n[\beta]_n\,\lambda^n x^{(n)}}{[(n)]!\,[\gamma][\gamma+p_2]\ldots[\gamma+(n+1)-(1)]},$$

we have (omitting for brevity α, β, λ, x, $p_1,\ldots$)

$$\frac{\mathfrak{F}(\gamma)}{\mathrm{F}(\gamma+p_1)} = \frac{(q^{\gamma-\alpha}-1)(q^{\gamma-\beta}-1)}{(q^{\gamma}-1)(q^{\gamma-\alpha-\beta}-1)},$$

which is identical with the corresponding ratio for Heine's series. It seems remarkable that the ratio for series having an infinite number of parameters should be identical with the ratio for series with only three parameters. Because of difference in the γ-factors of the two

types of series, I term these series *normal* and *abnormal* respectively.

A theorem due to M. J. M. Hill* relating to the sum of a finite number of terms of the series ${}_2F_1(\alpha, \beta; \gamma; 1)$ similarly has an exact counterpart in the case of the general series involving set numbers. A q-analogue of Hill's result was given by the present writer† some years ago. In this present paper the result for set numbers will be obtained. In the two papers to which reference is made the theorems were proved in a rather artificial way, involving prior knowledge of the form. In the analysis of the present paper the general theorem and its special case, given by Hill, arise naturally from the differential equation satisfied by the 'set' hypergeometric series.

In connexion with these theorems I refer here to a paper‡ 'Forms of Maclaurin's Theorem' in which the numbers (n) replace the integer n, and the relation of such forms to certain q-products and series of Jacobi is shown.

2. Notation

We write θ_r for the operative symbol $p_r x^{p_r} \dfrac{d}{d(x^{p_r})}$,

so that
$$\theta_r x^{(n)} = p_r x^{p_r} \frac{(n)}{p_r} x^{(n)-p_r} = (n)x^{(n)}.$$

Thus
$$\theta_1, \theta_2, \ldots, \theta_r, \ldots$$

are equivalent operative symbols, and may be denoted by a common symbol θ without distinguishing suffix.

Moreover,
$$q^\theta x^{(n)} \equiv q^{(n)} x^{(n)} \equiv (qx)^{(n)},$$

as is obvious on replacing q^θ by an exponential series of operators.

The operator $[\theta]$

We write
$$[\theta] \equiv (q^\theta - 1)/(q-1),$$

so that
$$[\theta]x^{(r)} \equiv \frac{q^{(r)}-1}{q-1} x^{(r)} \equiv [(r)]x^{(r)},$$

and
$$[\theta+\alpha]x^{(r)} \equiv [\alpha+(r)]x^{(r)}.$$

* M. J. M. Hill, *Proc. London Math. Soc.* (2) 5 (1907), 335–41.
† F. H. Jackson, *Messenger of Math.* 37 (1908), 123–6.
‡ F. H. Jackson, *Proc. London Math. Soc.* (2) 1 (1904), 351–5.

The operator Δ_r

We write
$$\Delta_r \equiv x^{-p_r}[\theta],$$
so that
$$\Delta_r x^{(n)} = [(n)]x^{(n)-p_r}.$$
The operators $\Delta_1, \Delta_2,..., \Delta_r,...$ are non-equivalent.

3. Generalization of Boole's operational equation

The operators Δ are related to the q^θ operators by the equation
$$x^{(n)}\Delta_1\Delta_2\dots\Delta_n = [\theta][\theta-(1)]...[\theta-(n-1)],$$
which is the generalization for set numbers of Boole's equation
$$x^nD^n = \theta(\theta-1)...(\theta-n+1).$$
The proof of this is given as Lemma I in the Appendix at the end of this paper, so as to avoid too much preliminary analysis interfering with the general results. A simpler basic analogue of Boole's equation was given by the writer some years ago.* When $|q| \to 1$ and the elements are units the equation reduces to Boole's operational equation.

4. The q^θ-difference-equations for set hypergeometric series

It is remarkable that the following *uncouth* equation
$$\{\lambda[\alpha][\beta]+\{\lambda q^\alpha[\beta+(1)]x^{(1)}+\lambda q^\beta[\alpha]x^{(1)}-[\gamma]\}\Delta_1+ \\ +\{\lambda x^{(2)}q^{\alpha+\beta+p_1}-q^\gamma x^{p_2}\}\Delta_1\Delta_2\}y = f_1(x)-f_2(x) \quad (1)$$
can be reduced very easily to the simpler form
$$x^{(1)}[\theta+\alpha][\theta+\beta]y-[\theta][\theta+\gamma-(1)]y = f_1(x)-f_2(x) \quad (2)$$
by using Lemma I of the Appendix. This corresponds to the well-known transformation for ordinary hypergeometric series
$$\{x(\theta+\alpha)(\theta+\beta)-\theta(\theta+\gamma-1)\}y = 0.$$

Both forms of equation are needed in the analysis, the arguments of which can be readily followed by comparison with the analysis of the ordinary differential equation satisfied by hypergeometric series of the second order. There is a third equation satisfied by the series $\mathfrak{F}$:
$$\left\{\lambda x^{(1)}\frac{[\theta+\alpha][\theta+\beta]}{[\theta+\gamma]}-[\theta]\right\}\mathfrak{F} = f_2(x)-f_3(x). \quad (3)$$
It is of abnormal form and appears in Lemma III of the Appendix.

* F. H. Jackson, *Messenger of Math.* 38 (1908), 57–61.

It is paradoxical that the normally formed series should satisfy the equation of abnormal form, while the abnormally formed series satisfy the equation of normal form.

5. Analysis of equation (2)

Substituting $$\sum_0^\infty A_n \lambda^n x^{(n)+\rho}$$

for y, we have

$$\sum_0^\infty \{\lambda A_n[\alpha+(n)+\rho][\beta+(n)+\rho]x^{(n)+\rho}-$$
$$-A_n[(n)+\rho][\gamma+(n)+\rho-p_1]x^{(n)+\rho-(1)}\} = f_1(x)-f_2(x). \quad (4)$$

The difference of the two infinite series cannot be made zero by choice of a suitable recurrence relation between the successive coefficients, except in certain obvious special cases: namely, the ordinary hypergeometric and the Heinean series. The difference f_1-f_2, however, vanishes when x is unity, provided that we choose as recurrence relation the following equation:

$$\lambda A_n[\alpha+(n)+\rho][\beta+(n)+\rho] = A_{n+1}[(n+1)+\rho][\gamma+(n+1)+\rho-p_1],$$

and also choose ρ such as to satisfy

$$A_n[\rho+(n)][\gamma+(n)+\rho-p_1]_{n=0} = 0.$$

Thus we have $\rho_1 = 0$, $\rho_2 = p_1-\gamma$ as principal roots. From the first of these we obtain

$$y = 1+\frac{[\alpha][\beta]}{[(1)][\gamma]}\lambda x^{(1)}+\ldots+$$
$$+\frac{[\alpha][\alpha+p_1]\ldots[\alpha+(n)][\beta]\ldots[\beta+(n)]\lambda^{n+1}x^{(n+1)}}{[(n+1)]!\,[\gamma][\gamma+p_2][\gamma+p_2+p_3]\cdots[\gamma+p_2+\cdots+p_{n+1}]}+\ldots. \quad (5)$$

This series is written at length to make clear the abnormality in the denominators. The γ-factors omit the element p_1.

The functional sign F is used to denote this series; the sign $\mathfrak{F}$ is used for the normal series

$$\sum \frac{[\alpha]_n[\beta]_n}{[n]!\,[\gamma]_n} x^{(n)}q^{n(\gamma-\alpha-\beta)}, \quad (6)$$

which arises naturally in the course of the analysis of the equations (1) and (2).

The preceding operations have, however, produced two series of abnormal form

$$\begin{aligned} f_1(x) &= \lambda[\alpha][\beta]\left\{1+\frac{[\alpha+p_1][\beta+p_1]}{[1][\gamma+p_2]}\lambda x^{p_1}+\ldots\right\}, \\ -f_2(x) &= -\lambda[\alpha][\beta]\left\{1+\frac{[\alpha+p_1][\beta+p_1]}{[1][\gamma+p_2]}\lambda x^{p_2}+\ldots\right\} \end{aligned} \tag{7}$$

differing in the indices of x, the indices of x in the general terms being respectively (n) and $(n+1)-(1)$.

In the Δ equation (1) the coefficient of the operator $\Delta_1\Delta_2$ vanishes when $x = 1$ and $\lambda = q^{\gamma-\alpha-\beta-p_1}$; also f_1-f_2 vanishes when $x = 1$.

Replacing $[\alpha]$ by $(q^\alpha-1)/(q-1)$, with similar replacement for $[\beta]$, $[\gamma]$, we obtain by elementary algebra

$$\left\{\mathrm{F} = \frac{(q-1)(q^{\gamma-\alpha-p_1}+q^{\gamma-\beta-p_1}-q^{\gamma-p_1}-1)}{(q^\alpha-1)(q^\beta-1)}\Delta_1\mathrm{F}\right\}_{x=1}, \tag{8}$$

in which $\mathrm{F} \equiv \mathrm{F}(\alpha,\beta;\gamma;\lambda,x,p_1\ldots)$ and $\lambda = q^{\gamma-\alpha-\beta-p_1}$.

From Lemma II (Appendix) we have

$$\mathfrak{F}(\alpha,\beta;\gamma-p_1;\lambda;x;p_1\ldots)-\mathrm{F}(\alpha,\beta;\gamma;\lambda;x;p_1\ldots) = \frac{q^{\gamma-p_1}}{[\gamma-p_1]}\Delta_1\mathrm{F}(x), \tag{9}$$

$$\mathfrak{F}(\alpha,\beta;\gamma-p_1;\lambda;x;p_1\ldots)-\mathrm{F}(\alpha,\beta;\gamma;\lambda;qx;p_1\ldots) = \frac{1}{[\gamma-p_1]}\Delta_1\mathrm{F}(x). \tag{10}$$

Notice that, in the F series of (10), qx appears on the left but not on the right side of the equation.

Elimination of $\{\Delta_1\mathrm{F}\}_{x=1}$

between (8) and (9) produces, after elementary algebra,

$$\frac{\mathfrak{F}(\alpha,\beta;\gamma-p_1;\lambda;1)}{\mathrm{F}(\alpha,\beta;\gamma;\lambda;1)} = \frac{(q^{\gamma-\alpha-p_1}-1)(q^{\gamma-\beta-p_1}-1)}{(q^{\gamma-p_1}-1)(q^{\gamma-\alpha-p_1}+q^{\gamma-\beta-p_1}-q^{\gamma-p_1}-1)}. \tag{11}$$

Elimination between (8) and (10) gives us

$$\frac{\mathfrak{F}(\alpha,\beta;\gamma-p_1;\lambda;1)}{\mathrm{F}(\alpha,\beta;\gamma,\lambda;q)} = \frac{(q^{\gamma-\alpha-p_1}-1)(q^{\gamma-\beta-p_1}-1)}{(q^{\gamma-p_1}-1)(q^{\gamma-\alpha-\beta-p_1}-1)}, \tag{12}$$

which again may be termed the abnormal and normal forms respectively. Replacing $\gamma-p_1$ by γ both in the factors of the series and in the solitary factor, we have

$$\frac{\mathfrak{F}(\alpha,\beta;\gamma,\mu;1)}{\mathrm{F}(\alpha,\beta;\gamma+p_1,\mu;q)} = \frac{[\gamma-\alpha][\gamma-\beta]}{[\gamma][\gamma-\alpha-\beta]} \quad (\mu = q^{\gamma-\alpha-\beta}). \tag{13}$$

This ratio is identical with that for Heine's series. Both (11) and (12) reduce to the same form when the p-elements are units, which is the case of Heine's series. Also, when $|q| \to 1$, the ratio (13) reduces to that for ordinary hypergeometric series. At this point it becomes manifest that theorem (13) cannot be used to obtain an analogue of the well-known expression for ${}_2F_1(\alpha, \beta; \gamma; 1)$ in gamma functions: the difficulty arises because $\mathrm{F}(\gamma+p_1) \neq \mathfrak{F}(\gamma+p_1)$. If, however, some simple value of a ratio could be found for $\mathfrak{F}(\gamma)/\mathrm{F}(\gamma)$, then a great advance would be possible.

So far I have not been able to find any simple expression for this ratio.

In Lemma III of the appendix it is shown that

$$\mathrm{F}(\gamma+p_1, x) = (1-q^\gamma)\{\mathfrak{F}(\gamma, x)+q^\gamma\mathfrak{F}(\gamma, qx)+q^{2\gamma}\mathfrak{F}(\gamma, q^2x)+\dots\}.$$

It is possible to write

$$\prod_0^\infty \frac{\mathfrak{F}\{\gamma+(r)\}}{\mathrm{F}\{\gamma+(r+1)\}} = \prod_{r=0}^\infty \frac{\{\gamma-\alpha+(r)\}\{\gamma-\beta+(r)\}}{\{\gamma+(r)\}\{\gamma-\alpha-\beta+(r)\}},$$

but until some simple expression for the ratio $\mathfrak{F}(\gamma)/\mathrm{F}(\gamma)$ can be found, there will be no complete analogue of

$$\mathrm{F}(\alpha, \beta; \gamma) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

I think it is unlikely, however, that any simple expression exists.

6. Finite series

If
$$\phi(x) = \sum_0^\infty A_r x^{(r)},$$

and all the elements $p_{n+1}, p_{n+2}, \dots$ are severally zero, then

$$\phi(x) = \sum_0^n A_r x^{(r)} + \{A_{n+1}+A_{n+2}+\dots\}x^{(n)}. \tag{14}$$

The indices of x do not increase after the $(n+1)$th term, and in hypergeometric series the coefficients $A_{n+1}, A_{n+2}, \dots$ form a series in geometrical progression, since we have

$$[\alpha+(n)] = [\alpha+(n+r)],$$

for all positive integers r.

If the series of coefficients $A_{n+1}, A_{n+2}, \dots$ form a convergent geometric series, we obtain

$$\phi(x) = \sum_0^n A_r x^{(r)} + A_{n+1}\frac{x^{(n)}}{1-R}, \tag{15}$$

in which $$R = A_{n+2}/A_{n+1}.$$

We see at once that this gives us a natural method of approach to Hill's and other analogous theorems.

We use

$$\sum_0^\infty \frac{[\alpha]_r[\beta]_r}{[r]!\,[\gamma]_r}\lambda^r = \frac{[\gamma-\alpha][\gamma-\beta]}{[\gamma][\gamma-\alpha-\beta]}\times\sum_0^\infty \frac{[\alpha]_r[\beta]_r}{[r]!\,[\gamma+p_1]}\lambda^r q^{(r)}. \tag{16}$$

In this it is only necessary to make $p_{n+1}, p_{n+2},\ldots$ (ad inf.) severally zero; then, by means of the simple result (15) of the preceding section applied to both the series on the right and left sides of the above equation, a generalization of Hill's theorem will follow after some elementary algebraic reduction.

In fact we have at once from (16)

$$\sum_0^n \frac{[\alpha]_r[\beta]_r}{[r]!\,[\gamma]_r}\lambda^r + \frac{[\alpha]_{n+1}[\beta]_{n+1}}{[n]!\,[n][\gamma]_{n+1}}\frac{\lambda^{n+1}}{1-R\lambda}$$

$$= \frac{[\gamma-\alpha][\gamma-\beta]}{[\gamma][\gamma-\alpha-\beta]}\left\{\sum_0^n \frac{[\alpha]_r[\beta]_r}{[r]!\,[\gamma+p_2]}\lambda^r q^{(r)} + \frac{[\alpha]_{n+1}[\beta]_{n+1}}{[n]!\,[n][\gamma+p_1]_n}\frac{\lambda^{n+1}q^{(n)}}{1-R\lambda}\right\}, \tag{17}$$

$$R\lambda = \frac{[\alpha+(n)][\beta+(n)]}{[n][\gamma+(n)]}q^{\gamma-\alpha-\beta},$$

$$1-R\lambda = -\frac{q^{(n)}+q^{(n)+\gamma}+q^{\gamma-\alpha-\beta}-q^{\gamma-\alpha+(n)}-q^{\gamma-\beta+(n)}-1}{(q^{(n)}-1)(q^{\gamma+(n)}-1)}.$$

Writing S_1, S_2 to represent the finite series, we have

$$S_1-\frac{[\gamma-\alpha][\gamma-\beta]}{[\gamma][\gamma-\alpha-\beta]}S_2 = \frac{[\alpha]_{n+1}[\beta]_{n+1}\lambda^{n+1}}{[n]!\,[n](1-R\lambda)}\left\{\frac{[\gamma-\alpha][\gamma-\beta]}{[\gamma][\gamma-\alpha-\beta]}\frac{q^{(n)}}{[\gamma+p_1]_n}-\frac{1}{[\gamma]_{n+1}}\right\}$$

$$= \frac{[\alpha]_{n+1}[\beta]_{n+1}}{[n]!\,[n][\gamma]_{n+1}}\left\{\frac{[\gamma-\alpha][\gamma-\beta]q^{(n)}}{[\gamma-\alpha-\beta][\gamma+(n)]}-1\right\}. \tag{18}$$

The expression in the brackets { } when multiplied out is

$$\frac{q^{(n)}+q^{\gamma+(n)}+q^{\gamma-\alpha-\beta}-q^{\gamma-\alpha+(n)}-q^{\gamma-\beta+(n)}-1}{(q^{\gamma-\alpha-\beta}-1)(q^{\gamma+(n)}-1)}.$$

The numerator is identical with that in the expression of

$$1-R\lambda;$$

so by simple cancelling of factors we obtain

$$S_1-\frac{[\gamma-\alpha][\gamma-\beta]}{[\gamma][\gamma-\alpha-\beta]}S_2 = \frac{[\alpha]_{n+1}[\beta]_{n+1}\lambda^{n+1}}{[n]!\,[\gamma]_{n+1}[\gamma-\alpha-\beta]}, \tag{19}$$

in which $\lambda = q^{\gamma-\alpha-\beta}$. Which, when $|q| \to 1$ and all elements are units from p_1 to p_n, gives the principal result of Hill's paper. When q is retained, we have the basic generalization given in a former paper by the present writer.

7. In conclusion it may be sufficient to note that, when $\alpha = -\beta$, we have

$$\mathfrak{F} = \sum_0^\infty \frac{(1-2q^{(n)}\cos\theta+q^{2(n)})!}{(1-q^{(n+1)})!\,(1-q^{\gamma+(n)})!} q^{(n+1)\gamma} x^{(n)}.$$

If the sequence of p be (1, 3, 5,...), the series becomes

$$\mathfrak{F} = \sum \frac{(1-2q^{n^2}\cos\theta+q^{2n^2})!}{(1-q^{(n+1)^2})!\,(1-q^{\gamma+n^2})} q^{(n+1)\gamma} x^{n^2},$$

in which the integers n are ordinary numbers.

It may be noted also that in the difference equations discussed in the preceding sections of the paper

$$f_1(x)-f_2(x)$$

reduces to a polynomial when the sequence of elements terminates.

8. Appendix

LEMMA I. *Generalization of Boole's operator equation.*

We have
$$q^\theta\phi(x) = \phi(qx),$$

$$\Delta_1\phi(x) = \frac{q^\theta-1}{x^{p_1}(q-1)}\phi(x) = \frac{\phi(qx)-\phi(x)}{x^{p_1}(q-1)},$$

$$\Delta_2\Delta_1\phi(x) = \left\{\frac{\phi(q^2x)-\phi(qx)}{q^{p_1}x^{p_1}(q-1)} - \frac{\phi(qx)-\phi(x)}{x^{p_1}(q-1)}\right\} \div x^{p_2}(q-1)$$

$$= \frac{\phi(q^2x)-(1+q^{p_1})\phi(qx)+q^{p_1}\phi(x)}{q^{p_1}x^{p_1+p_2}(q-1)^2}$$

$$= \frac{(q^\theta-1)(q^\theta-q^{p_1})}{q^{p_1}x^{(2)}(q-1)^2}\phi(x)$$

$$= \frac{[\theta][\theta-(1)]}{x^{(2)}}\phi(x).$$

Repetition, as simple algebra shows, gives

$$\Delta_3\Delta_2\Delta_1\phi(x) = \frac{(q^\theta-1)(q^\theta-q^{p_1})(q^\theta-q^{p_1+p_2})}{q^{2p_1+p_2}x^{(3)}}\phi(x$$

$$= \frac{[\theta][\theta-(1)][\theta-(2)]}{x^{(3)}}\phi(x).$$

The general case follows easily by induction:

$$x^{(n)}\Delta_n! = [\theta][\theta-(1)][\theta-(2)]...[\theta-(n-1)]. \tag{20}$$

LEMMA II.

Write (omitting unnecessary letters)

$$\mathfrak{F}(\alpha,\beta;\gamma;\mu;x) = \sum_0^\infty \frac{[\alpha]_n[\beta]_n}{[n]!\,[\gamma]_n}\mu^n x^{(n)} \quad (\mu = q^{\gamma-\alpha-\beta}),$$

the normal series, and

$$\mathrm{F}(\alpha,\beta;\gamma;\lambda;x)$$

$$= \sum_0^\infty \frac{[\alpha]_n[\beta]_n}{[n]!\,[\gamma][\gamma+p_2]...[\gamma+(n)-(1)]}\lambda^n x^{(n)} \quad (\lambda = q^{\gamma-\alpha-\beta-p_1}),$$

the abnormal series. Then, changing γ to $\gamma-p_1$ in the normal series, we find by simple algebra

$$\mathfrak{F}(\alpha,\beta;\gamma-p_1,\lambda,x)-\mathrm{F}(\alpha,\beta;\gamma;\lambda;x)$$

$$= \frac{\lambda x^{(1)}q^{\gamma-p_1}[\alpha][\beta]}{[\gamma][\gamma-p_1]}\left\{1+\frac{[\alpha+(1)][\beta+(1)]}{[1][\gamma+p_2]}\lambda x^{p_2}+...\right\}$$

$$= \frac{q^{\gamma-p_1}}{[\gamma-p_1]}\Delta_1\mathrm{F}.$$

Similarly, putting qx for x in the F series, we have

$$\mathfrak{F}(\alpha,\beta;\gamma-p_1;\lambda;x)-\mathrm{F}(\alpha,\beta;\gamma;\lambda;qx)$$

$$= \frac{\lambda x^{(1)}[\alpha][\beta]}{[\gamma][\gamma-p_1]}\left\{1+\frac{[\alpha+(1)][\beta+(1)]}{[1][\gamma+p_2]}\lambda x^{p_2}+...\right\}$$

and, omitting unnecessary letters,

$$\frac{\mathfrak{F}(\gamma-p_1,x)-\mathrm{F}(\gamma,x)}{\mathfrak{F}(\gamma-p_1,x)-\mathrm{F}(\gamma,qx)} = q^{\gamma-p_1},$$

an interesting equation, since x disappears from the quotient.

LEMMA III. *The difference equation for the normal series* $\mathfrak{F}$.

From Lemma II, by replacing $\gamma-p_1$ by γ, we obtain

$$(1-q^\gamma)\mathfrak{F}(\gamma,x) = \mathrm{F}(\gamma+p_1,x)-q^\gamma\mathrm{F}(\gamma+p_1,qx),$$

$$= (1-q^{\theta+\gamma})\mathrm{F}(\gamma+p_1,x).$$

From this we find

$$\mathrm{F}(\gamma+p_1,x)$$

$$= (1-q^\gamma)\{\mathfrak{F}(\gamma,x)+q^\gamma\mathfrak{F}(\gamma,qx)+q^{2\gamma}\mathfrak{F}(\gamma,q^2x)+q^{3\gamma}\mathfrak{F}(\gamma,q^3x)+...\}.$$

Now since $\mathbf{F}(\gamma, x)$ satisfies an equation

$$\{\lambda x^{(1)}[\theta+\alpha][\theta+\beta]-[\theta][\theta+\gamma-p_1]\}\mathbf{F}(\gamma, x)$$
$$= f_1(x)-f_2(x) \quad (\lambda = q^{\gamma-\alpha-\beta-p_1}),$$

we have, on replacing $\gamma-p_1$ by γ,

$$\{\mu x^{(1)}[\theta+\alpha][\theta+\beta]-[\theta][\theta+\gamma]\}\mathbf{F}(\gamma+p, x)$$
$$= f_3(x)-f_4(x) \quad (\mu = q^{\gamma-\alpha-\beta}),$$

and f_3, f_4 are obtained from f_1, f_2 by change of γ to $\gamma+p_1$.

Finally, since $$\mathbf{F}(\gamma+p_1, x) = \frac{[\gamma]}{[\gamma+\theta]}\mathfrak{F}(\gamma, x),$$

we have the equation for the normally formed series $\mathfrak{F}$ in a form

$$\frac{\mu x^{(1)}[\theta+\alpha][\theta+\beta]}{[\theta+\gamma]}\mathfrak{F}-[\theta]\mathfrak{F} = [\gamma]^{-1}\{f_3(x)-f_4(x)\}.$$

q^θ EQUATIONS AND FIBONACCI NUMBERS

By F. H. JACKSON (*Eastbourne*)

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1. Introduction

THE purpose of this paper is to discuss certain q-functional equations and their solutions in series of the form

$$\sum_{n=-\infty}^{n=+\infty} \phi(n)\, x^n, \tag{1}$$

in which $\phi(n)$ the coefficient of x^n is expressible in terms of 'Fibonacci numbers', that is, numbers of the sequence

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ldots,$$

in which
$$F_n = F_{n-1}+F_{n-2}.$$

It will be shown that in one case the coefficients may be transformed so that

$$\phi(n) = Aq^{n^2-n}\{(\omega_1)^{2n}+(\omega_2)^{2n}\}+Bq^{n^2-n}\{(\omega_1^{\frac{1}{2}})^{2n}+(\omega_2^{\frac{1}{2}})^{2n}\}, \tag{2}$$

where ω_1, ω_2 are the roots of $\omega^2 = \omega+1$, from which it follows that the Laurent series (1) can be expressed as the sum of four Jacobian functions $\mathfrak{J}(\omega x^2)$, namely

$$A\mathfrak{J}(\omega_1^2 x^2)+A\mathfrak{J}(\omega_2^2 x^2)+B\mathfrak{J}(\omega_1 x^2)+B\mathfrak{J}(\omega_2 x^2), \tag{3}$$

where

$$\mathfrak{J}(\omega_1^2 x^2) = \sum_{-\infty}^{+\infty} q^{n^2-n}\omega_1^{2n}x^{2n} = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-2}\omega_1^2 x^2)(1+q^{2n}\omega_1^{-2}x^{-2}).$$

2. Notation

In an article in this Journal*, certain functional equations—named 'q^θ equations' by the present writer—were discussed in the forms

$$\{q^{2\theta}+a_0 q^\theta+b_0\}y = 0, \tag{4}$$

$$\{q^{2\theta}+c_0 q^\theta+d_0\}\eta = 0, \tag{5}$$

$$\left\{q^{3\theta}-\frac{c_1}{a_0}(a_0 a_1-b_1)q^{2\theta}+d_1(a_0 a_1-b_1)q^\theta-\frac{a_1 d_1 b_0 d_0}{c_0}\right\}Y_1 = 0, \tag{6}$$

where
$$a_1 d_1/c_0 = b_1 c_1/a_0,$$

$$\{q^{4\theta}-a_2 c_2 q^{3\theta}+(c_1 c_2 b_2+a_1 a_2 d_2-b_2 d_2-\lambda)q^{2\theta}-\lambda a_0 c_0 q^\theta-\lambda b_0 d_0\}Y_2 = 0, \tag{7}$$

where
$$\lambda = b_2 c_2 d_1/c_0 = a_2 b_1 d_2/a_0.$$

* *Quart. J. of Math.* (Oxford), 11 (1940), 1–21.

In these equations the coefficients a_0, b_0,... are functional coefficients of an independent variable x; thus $a_0 = a(x)$, $a_n = a(q^n x)$,... and the operator q^θ denotes

$$\exp\left\{(\log q)x\frac{d}{dx}\right\}.$$

It follows that

$$q^\theta a(x) = a(qx) = a_1, \qquad q^{n\theta} a(x) = a_n.$$

Then, subject to these conditions, the product of solutions of equations (4) and (5) will satisfy equations (6) and (7) respectively. The usefulness of this brief subscript notation is obvious, when it is seen that the last term alone of equation (7) involves a product of six distinct functions. These theorems generalize for q-function theory certain theorems due to Appell and Goursat* for ordinary differential equations. Consider a pair of equations

$$\left\{q^{2\theta}+\frac{q^\theta}{qx}-\frac{1}{qx^2}\right\}y = 0, \tag{8}$$

$$\left\{q^{2\theta}+\frac{q^\theta}{qx}-\frac{1}{qx^2}\right\}\eta = 0, \tag{9}$$

in which $\quad a_0 = 1/qx = c_0, \qquad b_0 = -1/qx^2 = d_0. \qquad (10)$

The coefficients satisfy the condition necessary for the existence of an equation of the third order with solutions of form $y\times\eta$.

The equation (6), when the general coefficients are replaced by the particular values (10), reduces to

$$\{x^6q^{3\theta+6}-2x^4q^{2\theta+2}-2x^2q^\theta+1\}Y = 0. \tag{11}$$

To obtain solutions of the three equations we substitute series of form

$$\sum_{n=-\infty}^{+\infty} A_n x^n \tag{12}$$

for y and for η in equations (8) and (9), and

$$\sum_{n=-\infty}^{+\infty} B_n x^{2n} \tag{13}$$

for Y in the equation (11) of the third order.

In order that the series (12) may satisfy (8), we require that

$$\sum_{n=-\infty}^{+\infty} \{q^{2n}A_n x^n+q^{n-1}A_n x^{n-1}-q^{-1}A_n x^{n-2}\} = 0,$$

* P. Appell, *Comptes rendus*, tom. 91; E. Goursat, *Annales sci. de l'école normale*, ser. 2, tom. 12.

which can be valid only when there is the following recurrence relation among the undetermined coefficients

$$A_{n+2} = A_n q^{2n+1} + A_{n+1} q^{n+1}; \tag{14}$$

from which it is obvious that the coefficients A_n belong to a Fibonacci sequence of numbers.

If F_n denote the nth Fibonacci number, we obtain

$$A_n = q^{n(n-1)/2}\{F_{n-1} A_0 + F_n A_1\}, \tag{15}$$

which satisfies the recurrence equation (14). The sequence of coefficients is shown in the following table:

$$\begin{array}{ll}
A_0, & \\
A_1, & A_{-1} = q^1\{F_{-2} A_0 + F_{-1} A_1\}, \\
A_2 = q(A_0 + A_1), & A_{-2} = q^3\{F_{-3} A_0 + F_{-2} A_1\}, \\
A_3 = q^3(A_0 + 2A_1), & A_{-3} = q^6\{F_{-4} A_0 + F_{-3} A_1\}, \\
A_4 = q^6(2A_0 + 3A_1), & A_{-4} = q^{10}\{F_{-5} A_0 + F_{-4} A_1\}, \\
A_5 = q^{10}(3A_0 + 5A_1), & \cdots\cdots\cdots \\
\cdots\cdots & \cdots\cdots\cdots \\
A_n = q^{n(n-1)/2}(F_{n-1} A_0 + F_n A_1), & A_{-n} = q^{n(n+1)/2}\{F_{-n-1} A_0 + F_{-n} A_1\}.
\end{array} \tag{16}$$

The negative Fibonacci numbers are related to the positive by the equation

$$F_{-n} = (-1)^{n-1} F_n.$$

We have then

$$y = \sum_{n=-\infty}^{+\infty} q^{n(n-1)/2}\{F_{n-1} + \lambda F_n\} x^n, \tag{17}$$

on replacing A_0 by 1 and A_1 by an arbitrary λ. Similarly,

$$\eta = \sum_{n=-\infty}^{+\infty} q^{n(n-1)/2}\{F_{n-1} + \mu F_n\} x^n \tag{18}$$

satisfies the equation

$$\left\{q^{2\theta} + \frac{q^\theta}{qx} - \frac{1}{qx^2}\right\}\eta = 0.$$

3. Solution of (11)

$$\{x^6 q^{3\theta+6} - 2x^4 q^{2\theta+2} - 2x^2 q^\theta + 1\} Y = 0. \tag{19}$$

We substitute for Y the series

$$\sum_{n=-\infty}^{+\infty} B_n x^{2n}$$

and obtain

$$\sum_{n=-\infty}^{n=+\infty}\{B_n q^{6n+6}x^{2n+6}-2B_n q^{4n+2}x^{2n+4}-2B_n q^{2n}x^{2n+2}+B_n x^{2n}\}; \quad (20)$$

which can be made zero identically when the recurrence relation between four successive coefficients is

$$B_n q^{6n+6}-2B_{n+1}q^{4n+6}-2B_{n+2}q^{2n+4}+B_{n+3}\equiv 0.$$

If we attempted to solve this as a functional equation in finite differences in order to obtain an expression for B_n in terms of the first three coefficients B_0, B_1, B_2, the process would be tedious. The following *a priori* method suffices for the purpose of this paper.

By giving n the successive values 0, 1, 2,... we form the table of coefficients:

$$\left.\begin{aligned} B_0 &= 1\\ B_1 &= \mu,\\ B_2 &= \rho q^2,\\ B_3 &= (2\rho+2\mu-1)q^6,\\ B_4 &= (6\rho+3\mu-2)q^{12},\\ B_5 &= (15\rho+10\mu-6)q^{20},\\ B_6 &= (40\rho+24\mu-15)q^{30},\\ B_7 &= (104\rho+65\mu-40)q^{42},\\ &\ldots\ldots\ldots \end{aligned}\right\} \quad (21)$$

From the nature of the problem, which is that $y.\eta$ can be equated to Y conditionally on a suitable choice of arbitrary constants, we expect that the numbers in the B coefficients will be products of pairs of Fibonacci numbers. We notice that

$$B_7 = \{8.13\rho+5.13\mu-5.8\}q^{42},$$

$$B_6 = \{5.8\rho+3.8\mu-3.5\}q^{30},$$

in which the factors of products form an orderly set of Fibonacci numbers, thus

$$B_7 = \{F_6 F_7 \rho+\mu F_5 F_7-F_5 F_6\}q^{42},$$

and we conjecture that

$$B_n = \{F_{n-1}F_n\rho+F_{n-2}F_n\mu-F_{n-2}F_{n-1}\}q^{n^2-n}. \quad (22)$$

Trial shows that this does in fact satisfy the recurrence relation and we have

$$Y = \sum_{n=-\infty}^{+\infty} \{F_n F_{n-1}\rho + F_n F_{n-2}\mu - F_{n-1}F_{n-2}\} q^{n^2-n} x^{2n}. \qquad (23)$$

This expression may be transformed into Jacobi's series.

4. Jacobi's series

It is well known that the Fibonacci number F_n is formed as follows:

$$F_n = (\omega_1^n - \omega_2^n)/\sqrt{5}$$

in which $\omega_1 = (\sqrt{5}+1)/2 = -\omega_2^{-1}$,

so that Y may be expressed as

$$\sum_{-\infty}^{+\infty} \{q^{n^2-n}[a(\omega_1^n-\omega_2^n)(\omega_1^{n-1}-\omega_2^{n-1}) + b(\omega_1^n-\omega_2^n)(\omega_1^{n-2}-\omega_2^{n-2}) + \\ + c(\omega_1^{n-1}-\omega_2^{n-1})(\omega_1^{n-2}-\omega_2^{n-2})]\} x^{2n} \qquad (24)$$

by changing the form of the arbitrary constants.

We can multiply out the ω-factors and obtain

$$\left\{\begin{matrix} \omega_1^{2n} q^{n^2-n}(a\omega_1^{-1}+b\omega_1^{-2}+c\omega_1^{-3}) + \omega_2^{2n} q^{n^2-n}(a\omega_2^{-1}+b\omega_2^{-2}+c\omega_2^{-3}) + \\ + \omega_1^{n} q^{n^2-n}(a\omega_1^{-1}+b\omega_1^{-2}+c\omega_1^{-1}) + \omega_2^{n} q^{n^2-n}(a\omega_2^{-1}+b\omega_2^{-2}+c\omega_2^{-1}) \end{matrix}\right\} x^{2n}, \qquad (25)$$

owing to the property $\omega_1\omega_2 = -1$.

We can write this

$$(Aq^{n^2-n}\omega_1^{2n}x^{2n} + Bq^{n^2-n}\omega_2^{2n}x^{2n} + Bq^{n^2-n}\omega_1^{n}x^{2n} + Bq^{n^2-n}\omega_2^{n}x^{2n}), \qquad (26)$$

and have finally a solution

$$Y = A\mathfrak{J}(\omega_1^2 x^2) + A\mathfrak{J}(\omega_2^2 x^2) + B\mathfrak{J}(\omega_1 x^2) + B\mathfrak{J}(\omega_2 x^2), \qquad (27)$$

in which

$$\mathfrak{J}(\omega_1^2 x^2) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-2}x^2\omega_1^2)(1+q^{2n}x^{-2}\omega_1^{-2})$$
$$= \sum_{-\infty}^{+\infty} q^{n^2-n}\omega_1^{2n}x^{2n}.$$

I do not in this note compare this solution with the product of solutions y and η. It may be that some relations of interest exist when the series are equated after suitable choice of arbitrary constants. If such relations are found, I propose to consider them in another paper with more general equations

$$\{q^{2\theta} + \lambda x^a q^\theta + \mu x^{2a}\} y = 0, \qquad \{q^{2\theta} + \nu x^b q^\theta + \rho x^{2b}\}\eta = 0.$$

THE MODIFICATION OF AN INFINITE PRODUCT

By J. W. BRADSHAW (*Michigan*)

[Received 28 October 1941]

In an article by the author on 'modified series'* it was suggested that the type of procedure there applied to infinite series could be employed in connexion with infinite products. It is the purpose of this paper to develop this suggestion. The notion of a 'modified product' is not new; modified products have been noted by various authors.† But apparently no systematic procedure for obtaining such forms has been elaborated. Nor has the importance of the continued fraction in this connexion been stressed.

1. Definition. We assume as the notation for a convergent infinite product $\prod_{n=1}^{\infty}(1-p_n)$ and denote by $P_n = \prod_{k=1}^{n}(1-p_k)$ the product of the first n factors of the infinite product. If the last of these factors, $1-p_n$, is replaced by something different, say $1-q_n$, the product will be represented by

$$Q_n = (1-q_n)\prod_{k=1}^{n-1}(1-p_k), \tag{1}$$

and this form will be called a 'modified product'. Products which converge slowly may be thus 'modified' in order to find new forms that converge more rapidly. To this end we shall wish to find a function q_n of n such that Q_n and P_n approach the same limit and that the approach of Q_n to this limit will be more rapid than that of P_n. The first of these requirements will be met, if the condition

$$\lim_{n\to\infty}\frac{1-q_n}{1-p_n} = 1 \tag{2}$$

is satisfied. A way of getting at the second is to compare $1-Q_n/Q_{n-1}$

* J. W. Bradshaw, 'Modified series': *American Math. Monthly*, 46 (1939), 486.

† J. W. L. Glaisher. 'On the transformation of continued products into continued fractions': *Proc. London Math. Soc.* (1) 5 (1874), 78–88; J. W. Bradshaw, 'An infinite product for $\frac{1}{4}\pi$ derived from Gregory's series': *Quart. J. of Math.* 43 (1912), 378–9.

and $1-P_n/P_{n-1} = p_n$. The former should be of a lower order of magnitude than the latter. This may be formulated as the condition

$$\lim_{n\to\infty} n^\alpha \frac{(1-q_{n-1})-(1-q_n)(1-p_{n-1})}{p_n(1-q_{n-1})} = G \neq 0 \tag{3}$$

for some positive α.

2. Application to Wallis's product. The infinite product for $\frac{1}{2}\pi$ commonly ascribed to Wallis, viz. $\dfrac{2.2.4.4.6.6}{1.3.3.5.5.7}\ldots$, furnishes a good illustration, for it converges quite slowly. Combining the factors in pairs we should write

$$P_n = \prod_{k=1}^{n} \frac{2k.2k}{(2k-1)(2k+1)} = \prod_{k=1}^{n}\left(1+\frac{1}{4k^2-1}\right). \tag{4}$$

If for $1-q_n$ we choose a rational fraction whose numerator

$$\phi(n) = u_0 n^t + u_1 n^{t-1} + \ldots + u_t$$

and whose denominator

$$\psi(n) = v_0 n^t + v_1 n^{t-1} + \ldots + v_t$$

are of the same degree in n and have the same leading coefficient $u_0 = v_0$, the condition (2) will be satisfied. To satisfy the condition (3) we choose the coefficients so that

$$[4(n-1)^2-1]\phi(n-1)\psi(n)-4(n-1)^2\phi(n)\psi(n-1) \tag{5}$$

shall be of low degree in n.

By equating to zero the coefficients of successive powers of n in this expression we obtain the sequence of equations

$$\left.\begin{aligned} u_0-v_0 &= 0 \\ u_1-v_1 &= \tfrac{1}{4}v_0 \\ u_2-v_2 &= \tfrac{1}{4}v_1+\tfrac{5}{32}v_0 \\ u_3-v_3 &= \tfrac{1}{4}v_2+\tfrac{5}{32}v_1+\tfrac{11}{128}v_0 \\ u_4-v_4 &= \tfrac{1}{4}v_3+\tfrac{5}{32}v_2+\tfrac{11}{128}v_1+\tfrac{83}{2048}v_0 \\ u_5-v_5 &= \tfrac{1}{4}v_4+\tfrac{5}{32}v_3+\tfrac{11}{128}v_2+\tfrac{83}{2048}v_1+\tfrac{907}{8192}v_0 \end{aligned}\right\}. \tag{6}$$

By using in succession $t = 1, 2, 3,\ldots$ we obtain a succession of possible modifications $1-q_n$, viz.

$$\left.\begin{aligned} &\frac{8n-3}{8n-5} = 1+\frac{2}{8n-5}, \qquad \frac{64n^2-56n+15}{64n^2-72n+23} = 1+\frac{2}{8n-5+}\,\frac{1.3}{8n-4} \\ &\frac{512n^3-704n^2+464n-105}{512n^3-832n^2+592n-167} = 1+\frac{2}{8n-5+}\,\frac{1.3}{8n-4+}\,\frac{3.5}{8n-4} \end{aligned}\right\}. \tag{7}$$

The surmise that the continued fraction, through its successive convergents, will yield an unlimited number of these modifications is readily verified. We denote by $A_\rho(n)/B_\rho(n)$ the convergent of order ρ of the continued fraction

$$8n-5+\mathop{K}_{\rho=1}^{\infty}\frac{(2\rho-1)(2\rho+1)}{8n-4} \tag{8}$$

and for $1-q_n$ take $1+2\div\{A_\rho(n)/B_\rho(n)\}$. By the use of the reduction formulae for the continued fraction and mathematical induction we can then show that, if $\phi(n) = A_\rho(n)+2B_\rho(n)$ and $\psi(n) = A_\rho(n)$, the expression (5) will be independent of n for $\rho = 1, 2, 3,\ldots$.

While the original Wallis product converges slowly, this modified product, taking $n = 4$, $\rho = 4$, yields an approximation to $\frac{1}{2}\pi$ in which the error is less than 3 in the ninth decimal place.

3. Double modification. For values of ρ sufficiently large a modified continued fraction will serve our purpose even better than the continued fraction (8). By the term 'modified continued fraction' we mean

$$C_\rho/D_\rho = b_0+\frac{a_1}{b_1+}\cdots\frac{a_{\rho-1}}{b_{\rho-1}+}\frac{c_\rho}{d_\rho} \tag{9}$$

in which the partial quotient a_ρ/b_ρ has been replaced by c_ρ/d_ρ, chosen so as to give more rapid convergence.* For this purpose we take $c_\rho = a_\rho$ and for d_ρ a rational fraction whose coefficients are so determined that $d_\rho(d_{\rho-1}-b_{\rho-1})-a_\rho$ shall be of low degree in ρ. In this case it turns out that an appropriate value for d_ρ is twice any convergent of another continued fraction for which we take the index σ and whose convergents we distinguish by accents:

$$A'_\sigma/B'_\sigma = \frac{2\rho+4n-1}{2}+\frac{(2n-1)2n}{2\rho+1+}\cdots\frac{(2n-2+\sigma)(2n-1+\sigma)}{2\rho+1}. \tag{10}$$

We put $d_\rho = 2A'_\sigma/B'_\sigma$ and have then

$$\left.\begin{aligned} C_\rho &= 2\frac{A'_\sigma}{B'_\sigma}A_{\rho-1}+(2\rho-1)(2\rho+1)A_{\rho-2} \\ D_\rho &= 2\frac{A'_\sigma}{B'_\sigma}B_{\rho-1}+(2\rho-1)(2\rho+1)B_{\rho-2} \end{aligned}\right\}, \tag{11}$$

and our modification of the product replaces $1-p_n$ by $(C_\rho+2D_\rho)/C_\rho$.

* J. W. Bradshaw, 'Modified continued fractions', an article offered to the *American Math. Monthly*, but not yet accepted.

4. The function $F(n,\rho,\sigma)$. The doubly modified product is now a function of three indices; an approximation to $\frac{1}{2}\pi$ that is obtained in this way we denote by $F(n,\rho,\sigma)$. Replacing C_ρ and D_ρ by their values from (11), the formula for this function becomes

$$F(n,\rho,\sigma) = \frac{2.2.4.4...(2n-2)(2n-2)}{1.3.3.5...(2n-3)(2n-1)}\times$$
$$\times\frac{2A'_\sigma(A_{\rho-1}+2B_{\rho-1})+(2\rho-1)(2\rho+1)B'_\sigma(A_{\rho-2}+2B_{\rho-2})}{2A'_\sigma A_{\rho-1}+(2\rho-1)(2\rho+1)B'_\sigma A_{\rho-2}}$$
$$(\rho = 1,\, 2,\, 3,...). \quad (12)$$

In this formula we have

for ρ:

$$A_0 = 8n-5, \qquad B_0 = 1,$$
$$\left.\begin{aligned} A_\rho &= (8n-4)A_{\rho-1}+(2\rho-1)(2\rho+1)A_{\rho-2} \\ B_\rho &= (8n-4)B_{\rho-1}+(2\rho-1)(2\rho+1)B_{\rho-2} \end{aligned}\right\} \quad (\rho = 1,\, 2,\, 3,...);$$

for σ:

$$A'_0 = \tfrac{1}{2}(2\rho+4n-1), \qquad B'_0 = 1,$$
$$\left.\begin{aligned} A'_\sigma &= (2\rho+1)A'_{\sigma-1}+(2n-2+\sigma)(2n-1+\sigma)A'_{\sigma-2} \\ B'_\sigma &= (2\rho+1)B'_{\sigma-1}+(2n-2+\sigma)(2n-1+\sigma)B'_{\sigma-2} \end{aligned}\right\} \quad (\sigma = 1,\, 2,\, 3,...).$$

For $F(4,4,4)$ the formula yields a value which differs from $\frac{1}{2}\pi$ by less than 3 in the tenth decimal place.

5. A second modification. A different result will be obtained if only the factor $2n/(2n+1)$ is replaced; in other words, if for $1-q_n$ we take $2n\phi(n)/(2n-1)\psi(n)$. The continued fraction met in this way is

$$8n-1+\frac{1.3}{8n+}\,\frac{3.5}{8n+}...\frac{(2\rho-1)(2\rho+1)}{8n+}..., \quad (13)$$

and, if by $A_\rho(n)/B_\rho(n)$ we denote the convergent of order ρ of this continued fraction, then

$$\frac{\phi(n)}{\psi(n)} = \frac{A_\rho(n)}{A_\rho(n)+2B_\rho(n)}. \quad (14)$$

If this continued fraction is modified as in the previous case by replacing the denominator $8n$ in $(2\rho-1)(2\rho+1)/8n$ by a new continued fraction, this turns out to be twice

$$\tfrac{1}{2}(2\rho+4n+1)+\frac{2n(2n+1)}{2\rho+1+}...\frac{(2n-1+\sigma)(2n+\sigma)}{2\rho+1+}.... \quad (15)$$

The function of three indices at which we arrive in this way we denote by $G(n,\rho,\sigma)$, and the formula corresponding to (12) is

$$G(n,\rho,\sigma) = \frac{2.2.4.4...(2n-2)(2n-2)2n}{1.3.3.5...(2n-3)(2n-1)(2n-1)} \times$$
$$\times \frac{2A'_\sigma A_{\rho-1}+(2\rho-1)(2\rho+1)B'_\sigma A_{\rho-2}}{2A'_\sigma(A_{\rho-1}+2B_{\rho-1})+(2\rho-1)(2\rho+1)B'_\sigma(A_{\rho-2}+2B_{\rho-2})}. \quad (16)$$

6. Properties of F and G. Calculation of a considerable number of values of $F(n,\rho,\sigma)$ and $G(n,\rho,\sigma)$ from formulae (12) and (16) leads to the following surmises:

$$F(n,\rho,\sigma) = F(n+1,\rho,\sigma-2), \quad (17)$$
$$G(n,\rho,\sigma) = G(n+1,\rho,\sigma-2), \quad (18)$$
$$F(n,\rho,\sigma) = G(n,\rho,\sigma-1). \quad (19)$$

If these surmises are true, it may be contended that the use of the continued fractions (10) and (15) beyond A'_0/B'_0 has accomplished nothing of practical value; the same results could be obtained by an increase of n. However, this fact in itself is of some interest.

The illustrations we have discussed will, perhaps, suffice to suggest the usefulness of the notion of modified product and the importance of the continued fraction in this connexion.

THEOREMS CONCERNING MEAN VALUES OF ANALYTIC OR HARMONIC FUNCTIONS

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Introduction

1. In this paper we complete (with one reservation stated at the end of this section) our account of a body of work which has occupied us at intervals since 1924.* This work has been a piecemeal growth, and the logical order in which it stands is in some ways odd and anomalous. We can now give a more unified account of the main results (omitting some extensions which have been rather side-issues).

The critical proofs will also be much *shorter*. We cannot claim without reservation that they are *simpler*, because we allow ourselves to appeal to certain theorems of Littlewood and Paley which were not available until recently;† and the proofs of these theorems (of one in particular) are difficult. We could avoid appealing to these theorems if we chose. Parts of our analysis are quite independent of them; in others we could substitute *ad hoc* arguments such as we used before; and in one part,‡ where we thought that we should be compelled to use them, we now find them unnecessary. On balance it seems best to take advantage of them where we can.

The paper is not a mere revision of old work: it contains proofs of theorems stated before without proof, and one entirely new theorem (Theorem 8). In one respect, however, we do less than in our earlier papers, since we suppose throughout that the parameter r is greater than 1.§

We begin by a statement of the chief theorems to which we shall appeal.

* See in particular our papers **3**, **6**, **7**.

† They were stated without proof in 1931, and the first proofs published in 1937: see Littlewood and Paley, **10** and **11**.

‡ §§ 24–7. See **7**, 170, footnote †.

§ In §§ 20–1 we suppose only that $r \geqslant 1$. But a good many of the theorems are true for all positive r (and were proved so in our former papers).

Thus the proof of Theorem 3 of **7** remains incomplete: the case $0 < p < 1$ is still unaccounted for. Actually, this case is comparatively easy.

Theorems used

2. In what follows
$$f(z) = \sum_0^\infty c_n z^n \tag{2.1}$$
is an analytic function of $z = \rho e^{i\theta}$ regular for $\rho < 1$. The indices p, q, r, s are finite and satisfy
$$1 < p \leqslant 2 \leqslant q, \qquad r > 1, \qquad s > 1 \tag{2.2}$$
(except in §§ 20–1, where we allow r to be 1).

If $\phi(\theta)$ is any function of θ, then
$$\mathfrak{M}_r(\phi) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(\theta)|^r \, d\theta\right)^{1/r}. \tag{2.3}$$
In particular
$$\mathfrak{M}_r(f) = \mathfrak{M}_r(f, \rho) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^r \, d\theta\right)^{1/r}. \tag{2.4}$$

We use A, B, C as follows. A is, at each of its occurrences, a positive absolute constant; B a positive number depending only on the indices $p, q, \ldots$, or other parameters of the argument:* both A and B may differ at different occurrences even in the same formula. C is a positive number occurring in the hypothesis of a theorem and preserving its identity throughout the statement and proof of the theorem.

THEOREM A. *If*
$$\chi(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta - t)\psi(t) \, dt, \tag{2.5}$$
the Faltung $\mathfrak{F}(\phi, \psi)$
of ϕ *and* ψ, *and*
$$\alpha > 1, \qquad \beta > 1, \qquad \frac{1}{\alpha} + \frac{1}{\beta} > 1, \qquad \frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta} - 1, \tag{2.6}$$
then
$$\mathfrak{M}_\gamma(\chi) \leqslant \mathfrak{M}_\alpha(\phi)\mathfrak{M}_\beta(\psi). \tag{2.7}$$

This is a familiar inequality due to W. H. Young.† The inequality (and all others which we quote or prove) is to be interpreted as

* Which we shall indicate explicitly, by writing $B(p, \ldots)$, if there is any risk of confusion.

† See, for example, Hardy, Littlewood, and Pólya (**9**, 198–202) or Zygmund (**12**, 71).

meaning 'if the right-hand side is finite, then the left-hand side is also finite, and ...'.

THEOREM B. *If* $$\mathfrak{M}_r(f) \leqslant C \tag{2.8}$$
for $\rho < 1$, *then* f *can be expressed in the form*
$$f = f_1 + f_2, \tag{2.9}$$
where (i) f_1 *and* f_2 *are regular,* (ii) $f_1 \neq 0$ *and* $f_2 \neq 0$, *and* (iii)
$$\mathfrak{M}_r(f_1) \leqslant AC, \qquad \mathfrak{M}_r(f_2) \leqslant AC \tag{2.10}$$
for $\rho < 1$.

This theorem (which is actually true for all positive r) is proved in our paper **2**.* It is a simple corollary of a theorem of F. Riesz.

THEOREM C. *Suppose that* $S(\theta)$ *is the region bounded by the two tangents from the point* $e^{i\theta}$ *to the circle* $\rho = \rho_0 < 1$, *and the more distant arc of the circle between the points of contact, and that* $F(\theta)$ *is the upper bound of* $|f(z)|$ *in* $S(\theta)$. *Then*
$$\mathfrak{M}_r(F) \leqslant B(r, \rho_0) \lim_{\rho \to 1} \mathfrak{M}_r(f, \rho). \tag{2.11}$$

This theorem (which also is true for all positive r) is proved in our paper **5**.†

The form in which we have written (2.11) requires a word of explanation. The inequality says nothing unless the limit on the right-hand side is finite, i.e. unless $\mathfrak{M}_r(f)$ is bounded. In this case $f(z)$ has a 'boundary function' $f(e^{i\theta})$ of the class L^r: $f(\rho e^{i\theta}) \to f(e^{i\theta})$, when $\rho \to 1$, for almost all θ, and
$$\mathfrak{M}_r\{f(\rho e^{i\theta}) - f(e^{i\theta})\} \to 0,$$
so that $f(e^{i\theta})$ is also a 'strong limit' of $f(\rho e^{i\theta})$. Also
$$\lim_{\rho \to 1} \mathfrak{M}_r(f, \rho) = \mathfrak{M}_r\{f(e^{i\theta})\} = \mathfrak{M}_r(f, 1),$$
and (2.11) may be written in the simpler form
$$\mathfrak{M}_r(F) \leqslant B(r, \rho_0) \mathfrak{M}_r(f, 1). \tag{2.12}$$
But $f(z)$ may have a boundary function of L^r, defined as a radial limit, even when $\mathfrak{M}_r(f, \rho)$ is not bounded, so that we could not state (2.11) unconditionally in the form (2.12).

* Hardy and Littlewood, **2**, 207.

† Hardy and Littlewood, **5**, 114. The definition of $S(\theta)$ there is slightly different.

THEOREM D. *Suppose that* $1 < p \leqslant 2 \leqslant q$,* *and that*

$$c_0 = f(0) = 0. \tag{2.13}$$

Then
$$\lim_{\rho\to 1} \mathfrak{M}_p^p(f,\rho) \leqslant B \int_0^1 \int_{-\pi}^{\pi} (1-\rho)^{p-1} |f'(\rho e^{i\theta})|^p \, d\rho d\theta, \tag{2.14}$$

$$\int_0^1 \int_{-\pi}^{\pi} (1-\rho)^{q-1} |f'(\rho e^{i\theta})|^q \, d\rho d\theta \leqslant B \lim_{\rho\to 1} \mathfrak{M}_q^q(f,\rho). \tag{2.15}$$

When the limits are finite, they are equal to $\mathfrak{M}_p^p(f,1)$ and $\mathfrak{M}_q^q(f,1)$. We suppose $c_0 = 0$ (as we shall do in a number of later theorems) in order to avoid trivial complications. It is obvious, for example, that (2.14) could not be true without some such restriction, since the right-hand side is independent of c_0.

THEOREM E. *If* $c_0 = 0$, $r > 1$, *and*

$$g(\theta) = \left\{ \int_0^1 (1-\rho) |f'(\rho e^{i\theta})|^2 \, d\rho \right\}^{\frac{1}{2}}, \tag{2.16}$$

then
$$\mathfrak{M}_r(g) \leqslant B \lim_{\rho\to 1} \mathfrak{M}_r(f,\rho), \qquad \lim_{\rho\to 1} \mathfrak{M}_r(f,\rho) \leqslant B\mathfrak{M}_r(g). \tag{2.17}$$

There is naturally the same gloss about $\mathfrak{M}_r(f,1)$. Theorems D and E contain Theorems 5–7 of the second Littlewood-Paley paper (Theorems 1–3 of the first).

THEOREM F. *If*

$$u(z) = u(\rho,\theta) = \sum_{-\infty}^{\infty} c_n \rho^{|n|} e^{ni\theta} \tag{2.18}$$

is a harmonic function of z regular for $\rho < 1$, *and*

$$\mathfrak{M}_r(u) \leqslant C; \tag{2.19}$$

and if we write

$$u(\rho,\theta) = \sum_0^{\infty} + \sum_{-\infty}^{-1} = \sum_0^{\infty} c_n \rho^n e^{ni\theta} + \sum_1^{\infty} c_{-n} \rho^n e^{-ni\theta} = u_1(\rho,\theta) + u_2(\rho,\theta); \tag{2.20}$$

then
$$\mathfrak{M}_r(u_1) \leqslant BC, \qquad \mathfrak{M}_r(u_2) \leqslant BC. \tag{2.21}$$

This is equivalent to M. Riesz's theorem concerning conjugate functions of the class L^r.† We shall use the theorem once only (in § 17), and then u will be a (harmonic) polynomial. In this case

$$\mathfrak{M}_r(u,\rho) \leqslant \mathfrak{M}_r(u,1),$$

* As in (2.2): but we repeat the inequalities for greater emphasis.

† See Hardy and Littlewood (**1**) for further explanations.

and we can take $C = \mathfrak{M}_r(u, 1)$, when

$$\mathfrak{M}_r(u_1, 1) = \lim_{\rho\to 1} \mathfrak{M}_r(u_1, \rho) \leqslant BC = B\mathfrak{M}_r(u, 1). \tag{2.22}$$

It is in this form that we shall actually use the theorem.

Some lemmas

3. We shall also require the following lemmas. Lemma γ, in particular, enables us to avoid many trivial complications, and we shall use it repeatedly.

LEMMA α. *If*

$$f_1(z) = \int_0^z f(u)\,du \tag{3.1}$$

and $k \geqslant 1$, *then*

$$\mathfrak{M}_k(f_1, \rho) \leqslant \rho\mathfrak{M}_k(f, \rho). \tag{3.2}$$

For
$$|f_1(\rho e^{i\theta})| = \left|e^{i\theta}\int_0^\rho f(\sigma e^{i\theta})\,d\sigma\right| \leqslant \int_0^\rho |f(\sigma e^{i\theta})|\,d\sigma,$$

$$\mathfrak{M}_k(f_1) \leqslant \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\left\{\int_0^\rho |f(\sigma e^{i\theta})|\,d\sigma\right\}^k\right]^{1/k} \leqslant \int_0^\rho d\sigma\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\sigma e^{i\theta})|^k\,d\theta\right\}^{1/k},$$

by 'Minkowski's inequality';* and this is

$$\int_0^\rho \mathfrak{M}_k(f, \sigma)\,d\sigma \leqslant \rho\mathfrak{M}_k(f, \rho),$$

because $\mathfrak{M}_k(f, \rho)$ increases with ρ.

LEMMA β. *If* $k \geqslant 0$, $z^{-k}f(z)$ *is regular at the origin*, $r > 0$† *and* $0 < \lambda < 1$, *then*

$$\mathfrak{M}_r(f, \rho) \leqslant \rho^{k(1-\lambda)}\mathfrak{M}_r(f, \rho^\lambda). \tag{3.3}$$

For $0 < \rho < \rho^\lambda < 1$ and

$$\rho^{-k}\mathfrak{M}_r(f, \rho) = \mathfrak{M}_r\{z^{-k}f(z)\}$$

increases with ρ, so that

$$\rho^{-k}\mathfrak{M}_r(f, \rho) \leqslant \rho^{-\lambda k}\mathfrak{M}_r(f, \rho^\lambda).$$

LEMMA γ. *Suppose that a and b are real and c and r positive, and that*

$$J = \int_0^1 (1-\rho)^a\rho^{-b}\mathfrak{M}_r^c(f, \rho)\,d\rho \tag{3.4}$$

* See Hardy, Littlewood, and Pólya, **9**, 148 (Theorem 202).

† We are concerned only with the case $r \geqslant 1$, but the proof is independent of this hypothesis.

is convergent at the origin. Then

$$J \leqslant B \int_0^1 (1-\rho)^a \mathfrak{M}_r^c(f,\rho)\,d\rho, \tag{3.5}$$

where B depends on a, b and c.

If $b \leqslant 0$, there is nothing to prove. If $b > 0$, let k be the least integer for which $b < ck+1$, and $0 < \lambda < 1$. Then $z^{-k}f(z)$ is regular at the origin, and

$$J \leqslant \int_0^1 (1-\rho)^a \rho^{ck(1-\lambda)-b} \mathfrak{M}_r^c(f,\rho^\lambda)\,d\rho,$$

by Lemma β. If we take

$$\lambda = 1-\frac{b}{ck+1}$$

(when $0 < \lambda < 1$), and put $\rho^\lambda = \sigma$, we obtain

$$J \leqslant \frac{1}{\lambda} \int_0^1 (1-\sigma^{1/\lambda})^a \mathfrak{M}_r^c(f,\sigma)\,d\sigma. \tag{3.6}$$

But
$$1 \leqslant \frac{1-\sigma^{1/\lambda}}{1-\sigma} \leqslant \frac{1}{\lambda},$$

so that (3.6) is equivalent to (3.5).

Lemma δ. *If $0 \leqslant \rho \leqslant 1$ and n is a positive integer greater than 1, then*

$$\int_{-\pi}^{\pi} \left|\frac{1-z^n}{1-z}\right| d\theta < A \log n.$$

For the integral increases with ρ, and therefore

$$\int_{-\pi}^{\pi} \left|\frac{1-z^n}{1-z}\right| d\theta \leqslant \int_{-\pi}^{\pi} \left|\frac{1-e^{ni\theta}}{1-e^{i\theta}}\right| d\theta = 2\int_0^{\pi} \frac{|\sin \frac{1}{2}n\theta|}{\sin\frac{1}{2}\theta}\,d\theta$$

$$\leqslant 2\int_0^{1/n} n\,d\theta + 2\int_{1/n}^{\pi} \frac{d\theta}{\sin\frac{1}{2}\theta} \leqslant 2+2\pi \int_{1/n}^{\pi} \frac{d\theta}{\theta} < 2+2\pi \log n.$$

The inequality is of course familiar.

Preliminary theorems

4. THEOREM 1.* *If* $1 < r < s$ *and*

$$\mathfrak{M}_r(f) \leqslant C, \tag{4.1}$$

then
$$\mathfrak{M}_s(f) \leqslant BC(1-\rho)^{-\left(\frac{1}{r}-\frac{1}{s}\right)} \tag{4.2}$$

and
$$\mathfrak{M}(f) = \mathfrak{M}_\infty(f) \leqslant BC(1-\rho)^{-1/r}. \tag{4.3}$$

Here $\mathfrak{M}(f)$ is the maximum modulus of $|f(z)|$ for $|z| = \rho$.

After Theorem B, we can write $f = f_1+f_2$, where $f_1 \neq 0$, $f_2 \neq 0$, $\mathfrak{M}_r(f_1) \leqslant AC$ and $\mathfrak{M}_r(f_2) \leqslant AC$ for $\rho < 1$. If we have proved (4.2) and (4.3) for f_1 and f_2, then

$$\mathfrak{M}(f) \leqslant \mathfrak{M}(f_1)+\mathfrak{M}(f_2) \leqslant BC(1-\rho)^{-1/r}$$

and
$$\mathfrak{M}_s(f) \leqslant \mathfrak{M}_s(f_1)+\mathfrak{M}_s(f_2) \leqslant BC(1-\rho)^{-\left(\frac{1}{r}-\frac{1}{s}\right)}$$

(the last by Minkowski's inequality†). It is therefore sufficient to prove the theorem for an f without zeros in $\rho < 1$.

If f has no zeros, we can write $f = \phi^{2/r}$, where ϕ is regular in $\rho < 1$. Then
$$\mathfrak{M}_2^2(\phi) = \mathfrak{M}_r^r(f) \leqslant C^r, \qquad \mathfrak{M}_s^s(f) = \mathfrak{M}_{2s/r}^{2s/r}(\phi).$$

If the theorem has been proved for $r = 2$, then (applying that case to ϕ)

$$\mathfrak{M}_s^s(f) = \mathfrak{M}_{2s/r}^{2s/r}(\phi) \leqslant BC^{\frac{r}{2}\cdot\frac{2s}{r}}(1-\rho)^{-\frac{2s}{r}\left(\frac{1}{2}-\frac{r}{2s}\right)} = BC^s(1-\rho)^{-\left(\frac{s}{r}-1\right)},$$

which is (4.2); and

$$\mathfrak{M}(f) = \mathfrak{M}^{2/r}(\phi) \leqslant BC(1-\rho)^{-\frac{1}{2}(2/r)} = BC(1-\rho)^{-1/r},$$

which is (4.3), It is therefore sufficient to prove the theorem for $r = 2$.

When $r = 2$, we have

$$\sum |c_n|^2 = \mathfrak{M}_2^2(f) \leqslant C^2;$$

so that

$$\mathfrak{M}(f) \leqslant \sum |c_n|\rho^n \leqslant \left(\sum |c_n|^2 \sum \rho^{2n}\right)^{\frac{1}{2}} \leqslant C(1-\rho)^{-\frac{1}{2}}.$$

Also

$$\mathfrak{M}_s^s(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^s\,d\theta \leqslant \mathfrak{M}^{s-2}(f)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\,d\theta$$
$$\leqslant \{C(1-\rho)^{-\frac{1}{2}}\}^{s-2}C^2 = \left\{C(1-\rho)^{-\left(\frac{1}{2}-\frac{1}{s}\right)}\right\}^s.$$

These are the inequalities required.

* See Hardy and Littlewood (**4**, 623–5, and **6**, 406–7). We repeat the original proof, which works for $0 < r < s$.

† In one of its more usual forms, e.g. Theorem 198 of Hardy, Littlewood, and Pólya, **9**, 146.

We shall sometimes require variants of Theorem 1 in which the C of (4.1) is replaced by a $c(\rho)$ which tends steadily to infinity when $\rho \to 1$. If then we fix a σ between 0 and 1, and apply Theorem 1 to $f(\sigma z) = f(\sigma\rho e^{i\theta})$, we obtain, for example,

$$\mathfrak{M}(f, \sigma\rho) \leqslant B(1-\rho)^{-1/r}\mathfrak{M}_r(f, \sigma) \leqslant B(1-\rho)^{-1/r}c(\sigma),$$

and so $\quad \mathfrak{M}(f, \sigma^2) \leqslant B(1-\sigma)^{-1/r}c(\sigma) \leqslant B(1-\sigma^2)^{-1/r}c(\sigma).$

Finally, replacing σ^2 by ρ, we obtain

THEOREM 2. *If $\mathfrak{M}_r(f, \rho) \leqslant c(\rho)$, then*

$$\mathfrak{M}_s(f) \leqslant B(1-\rho)^{-\left(\frac{1}{r}-\frac{1}{s}\right)}c(\rho^{\frac{1}{2}}), \qquad \mathfrak{M}(f) \leqslant B(1-\rho)^{-\frac{1}{r}}c(\rho^{\frac{1}{2}}). \tag{4.4}$$

The most important cases are those in which (i) $c(\rho)$ is $\mathfrak{M}_r(f, \rho)$ itself, and (ii) $c(\rho) = (1-\rho)^{-a}$, where $a > 0$. In the latter case we can replace $c(\rho^{\frac{1}{2}})$, in (4.4), by $c(\rho)$. Thus, if $\mathfrak{M}_r(f)$ is of order $(1-\rho)^{-a}$, $\mathfrak{M}(f)$ is of order $(1-\rho)^{-a-1/r}$ at most.

5. THEOREM 3.* *If $\mathfrak{M}_r(f) \leqslant C$, then*

$$\mathfrak{M}_r(f') \leqslant \frac{BC}{1-\rho}. \tag{5.1}$$

We have $$f'(z) = \frac{1}{2\pi i}\int_{D(\theta,\rho)} \frac{f(u)}{(u-z)^2}\,du,$$

where $D(\phi, \rho)$ or $D(\theta)$ is a curve inside $\rho < 1$ and round $u = z = \rho e^{i\theta}$. We distinguish two cases.

(i) If $\rho < \frac{1}{4}$, we take $D(\theta)$ to be the circle $|u| = \frac{1}{2}$. Then $|u-z| > \frac{1}{4}$ on $D(\theta)$, and so

$$|f'(z)| \leqslant 16\mathfrak{M}_1(f, \tfrac{1}{2}) \leqslant 16\mathfrak{M}_r(f, \tfrac{1}{2}) \leqslant 16C,$$

and *a fortiori* $$\mathfrak{M}_r(f', \rho) \leqslant 16C \leqslant \frac{BC}{1-\rho}.$$

(ii) If $\rho \geqslant \frac{1}{4}$, we take $D(\theta)$ to be the circle whose centre is $u = z = \rho e^{i\theta}$ and which passes through $u = \rho^{\frac{1}{2}}e^{i\theta}$. The radius of this circle is $\rho^{\frac{1}{2}}-\rho$, and lies between two numbers $A(1-\rho)$, so that the circle is inside a region of the type of the $S(\theta)$ of Theorem C. Hence

$$|f'(z)| \leqslant \frac{AF(\theta)}{1-\rho}$$

* Actually $\mathfrak{M}_r(f') = o\{(1-r)^{-1}\}$, but we do not need this here.

Most of the content of Theorems 3–6 is to be found in Hardy and Littlewood, **6**, **430** et seq. The results proved there are in some ways a little less precise, but are proved for all positive r.

and $$\mathfrak{M}_r(f') \leqslant \frac{B}{1-\rho}\mathfrak{M}_r(F) \leqslant \frac{BC}{1-\rho},$$
by Theorem C.

THEOREM 4. *If $\mathfrak{M}_r(f) \leqslant c(\rho)$, then*
$$\mathfrak{M}_r(f') \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{\rho^{\frac{1}{2}}(1-\rho)}.* \tag{5.2}$$

Applying Theorem 3 to $f(\sigma z)$, we obtain
$$\mathfrak{M}_r\{\sigma f'(\sigma\rho e^{i\theta})\} \leqslant \frac{B\mathfrak{M}_r\{f(\sigma e^{i\theta})\}}{1-\rho} \leqslant \frac{Bc(\sigma)}{1-\rho}.$$

Hence $$\mathfrak{M}_r(f', \rho^2) \leqslant \frac{Bc(\rho)}{\rho(1-\rho)},$$
which is equivalent to (5.2).

Fractional derivatives

6. We shall require similar results for $f^{\beta}(z)$, the βth derivative, of non-integral order, of $f(z)$. There are various definitions of these derivatives, which are not quite equivalent for our present purposes. We take first the definition (which we distinguish from our later definition by putting the β in brackets)
$$f^{(\beta)}(z) = \frac{\Gamma(1+\beta)}{2\pi i}\int\limits_{D(z)} (u-z)^{-1-\beta}f(u)\,du, \tag{6.1}$$
where $D(z)$ is a loop from the origin lying inside the unit circle and encircling $u = z$ in the positive direction,
$$\begin{aligned}(u-z)^{-1-\beta} &= \exp\{-(1+\beta)\log(u-z)\} \\ &= \exp[-(1+\beta)\{\log|u-z|+i\,\mathrm{am}(u-z)\}],\end{aligned}$$
and $\mathrm{am}(u-z) = \mathrm{am}\,z = \theta$ at the point where $D(z)$ cuts the radius vector through $u = z$.

The formula (6.1) defines $f^{(\beta)}(z)$ for all real (or complex) values of β except negative integral values. When β is negative, it may be replaced by
$$f^{(\beta)}(z) = \frac{1}{\Gamma(-\beta)}\int\limits_0^z (z-u)^{-1-\beta}f(u)\,du, \tag{6.2}$$
where the path of integration is rectilinear. This formula is significant

* The factor $\rho^{\frac{1}{2}}$ in the denominator is essential; for $c(\rho)$ can be small for small ρ if $f(0) = 0$, while $\mathfrak{M}_r(f')$ is usually not small.

except for $\beta = 0, 1, \ldots$ (so that the two together cover all values of β). In any case

$$f^{(\beta)}(z) = \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} c_n z^{n-\beta}, \tag{6.3}$$

and $z^\beta f^{(\beta)}(z)$ is regular for $\rho < 1$. When β is a positive integer, $f^{(\beta)}(z)$ is an ordinary derivative of $f(z)$, and, when β is a negative integer,

$$f_{(-\beta)}(z) = f^{(\beta)}(z) \tag{6.4}$$

is an ordinary (repeated) integral of $f(z)$.

We can still define $f^{(\beta)}(z)$ by (6.1) or (6.2) when $f(z)$ is of the form

$$f(z) = \sum c_n z^{n+\lambda}$$

and $\lambda > -1$. We have then

$$f^{(\beta)}(z) = \sum \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\beta)} c_n z^{n+\lambda-\beta},$$

instead of (6.3). The derivatives and integrals thus defined obey the ordinary operational laws

$$(f^{(\beta)})^{(\gamma)} = (f^{(\gamma)})^{(\beta)} = f^{(\beta+\gamma)}.$$

The important functions are $f^{(\beta)}(z)$ and $f_{(\beta)}(z)$ with $0 < \beta < 1$.

7. Theorem 5. *If $\mathfrak{M}_r(f) \leqslant C$ and $\beta > 0$, then*

$$\mathfrak{M}_r(f^{(\beta)}) \leqslant \frac{BC}{\rho^\beta(1-\rho)^\beta}. \tag{7.1}$$

We use the formula (6.1), and again distinguish two cases.

(i) If $\rho < \frac{1}{4}$, we take $D(z)$ to be the circle through the origin whose centre is $u = z$. This circle lies within a region $S(\theta)$, and

$$|f^{(\beta)}(z)| \leqslant \frac{BF(\theta)}{\rho^\beta} \leqslant \frac{BF(\theta)}{\rho^\beta(1-\rho)^\beta}.$$

(ii) If $\frac{1}{4} \leqslant \rho < 1$, we take $D(z)$ to be the contour formed by (*a*) the circle used in the proof of case (ii) of Theorem 3, and (*b*) the straight line from the origin to the nearest point on the circle, described twice in opposite directions: this contour also lies within an $S(\theta)$. The modulus of the contribution of (*a*) does not exceed

$$\frac{BF(\theta)}{(\rho^{\frac{1}{2}}-\rho)^\beta} \leqslant \frac{BF(\theta)}{\rho^\beta(1-\rho)^\beta},$$

and that of (*b*) does not exceed

$$BF(\theta)\int_0^{2\rho-\rho^{\frac{1}{2}}} \frac{d\sigma}{(\rho-\sigma)^{1+\beta}} \leqslant \frac{BF(\theta)}{(\rho^{\frac{1}{2}}-\rho)^{\beta}} \leqslant \frac{BF(\theta)}{\rho^{\beta}(1-\rho)^{\beta}}.$$

Hence, in any case,

$$|f^{(\beta)}(z)| \leqslant \frac{BF(\theta)}{\rho^{\beta}(1-\rho)^{\beta}};$$

and
$$\mathfrak{M}_r(f^{(\beta)}) \leqslant \frac{B\mathfrak{M}_r(F)}{\rho^{\beta}(1-\rho)^{\beta}} \leqslant \frac{BC}{\rho^{\beta}(1-\rho)^{\beta}},$$

by Theorem C.

THEOREM 6. *If* $\mathfrak{M}_r(f) \leqslant c(\rho)$ *and* $\beta > 0$, *then*

$$\mathfrak{M}_r(f^{(\beta)}) \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{\rho^{\beta}(1-\rho)^{\beta}}. \tag{7.2}$$

Applying Theorem 5 to $f(\sigma z)$, we obtain

$$\mathfrak{M}_r\{\sigma^{\beta} f^{(\beta)}(\sigma\rho e^{i\theta})\} \leqslant \frac{Bc(\sigma)}{\rho^{\beta}(1-\rho)^{\beta}}.$$

Hence
$$\mathfrak{M}_r\{f^{(\beta)}(\rho^2 e^{i\theta})\} \leqslant \frac{Bc(\rho)}{\rho^{2\beta}(1-\rho)^{\beta}} \leqslant \frac{Bc(\rho)}{\rho^{2\beta}(1-\rho^2)^{\beta}},$$

which is equivalent to (7.2).

It will be observed that Theorems 3 and 4 are not included in Theorems 5 and 6, being sharper than the results of taking $\beta = 1$ in those theorems. This is natural because $f^{(\beta)}(z)$ has generally a singularity at the origin, which disappears when $\beta = 1$.

8. Our second definition of the derivative of order β applies only when
$$f(0) = c_0 = 0. \tag{8.1}$$

It is
$$f^{\beta}(z) = i^{\beta} \sum_1^{\infty} n^{\beta} c_n z^n. \tag{8.2}$$

We also write
$$f_{\beta}(z) = f^{-\beta}(z). \tag{8.3}$$

These definitions can be used for all real (or complex) β, and it is plain that
$$(f^{\beta})^{\gamma} = (f^{\gamma})^{\beta} = f^{\beta+\gamma}$$
for all β and γ.

The difference between the two systems of definitions is roughly that between differentiation (or integration) with respect to $z = \rho e^{i\theta}$ and with respect to θ. The definitions of this section, which require

c_0 to be 0, are the most convenient in the theory of Fourier series. If $\alpha = -\beta > 0$ and $\rho < 1$, then

$$\frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\theta} (\theta - t)^{\alpha-1} f(\rho e^{it})\, dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\theta} (\theta - t)^{\alpha-1} \sum_1^\infty c_n \rho^n e^{nit}\, dt = \sum_1^\infty (ni)^{-\alpha} c_n \rho^n e^{ni\theta} = f_\alpha(\rho e^{i\theta});$$

and this equation can be extended, if properly interpreted, to the case $\rho = 1$. Thus $f_\alpha(z)$ is effectively the αth integral of $f(z)$, as defined by Weyl.

THEOREM 7. *If* $0 < \beta < 1$* *and*

$$\mathfrak{M}_r(f) \leqslant c(\rho), \tag{8.4}$$

then
$$\mathfrak{M}_r(f^\beta) \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}. \tag{8.5}$$

Here there is no factor $\rho^{-\beta}$.

If we write for the moment

$$g = z^\beta f^{(\beta)} = \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} c_n z^n,$$

then
$$\mathfrak{M}_r(g) \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}, \tag{8.6}$$

by Theorem 6: and we shall prove that $i^{-\beta} f^\beta - g$ satisfies a similar inequality. Now

$$n^\beta = \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} + \frac{B}{n+1} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} + u_n,$$

where
$$|u_n| \leqslant Bn^{\beta-2}.$$

Hence
$$i^{-\beta} f^\beta - g = B\phi + \psi, \tag{8.7}$$

where
$$\phi = \sum \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} \frac{c_n}{n+1} z^n, \qquad \psi = \sum u_n c_n z^n;$$

and it is sufficient to prove that $\mathfrak{M}_r(\phi)$ and $\mathfrak{M}_r(\psi)$ satisfy inequalities of the type (8.5).

Now ϕ is related to

$$\sum \frac{c_n}{n+1} z^n = \frac{1}{z} \int_0^z f(u)\, du = \frac{f_1(z)}{z},$$

* From this point onwards we confine our attention to this case.

as g is to f. Hence, applying (8.6) to f_1/z, and using Lemma α, we obtain

$$\mathfrak{M}_r(\phi) \leqslant \frac{B}{(1-\rho)^\beta}\mathfrak{M}_r\left(\frac{f_1}{z}, \rho^{\frac{1}{2}}\right) \leqslant \frac{B}{(1-\rho)^\beta}\mathfrak{M}_r(f, \rho^{\frac{1}{2}}) \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}. \quad (8.8)$$

Finally, $$|\psi| \leqslant B\sum |c_n| n^{\beta-2}\rho^n.$$

But $$|c_n|\rho^n \leqslant \mathfrak{M}_1(f) \leqslant \mathfrak{M}_r(f);$$

and so (since $\beta-2 < -1$)

$$\mathfrak{M}_r(\psi) \leqslant \mathfrak{M}(\psi) \leqslant \mathfrak{M}_r(f)\sum n^{\beta-2} = B\mathfrak{M}_r(f) \leqslant \frac{B\mathfrak{M}_r(f,\rho^{\frac{1}{2}})}{(1-\rho)^\beta} \leqslant \frac{Bc(\rho^{\frac{1}{2}})}{(1-\rho)^\beta}. \quad (8.9)$$

The theorem now follows from (8.7), (8.8), and (8.9).

9. Theorem 8. *If $r > 1$, $s > 1$, $b < 1$, and $f(0) = 0$, then*

$$B \leqslant \frac{\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\, d\rho}{\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\, d\rho} \leqslant B. \quad (9.1)$$

In particular, when $s = r$,

$$B \leqslant \frac{\int_0^1\int_{-\pi}^{\pi} (1-\rho)^{-b}|f|^r\, d\rho d\theta}{\int_0^1\int_{-\pi}^{\pi} (1-\rho)^{r-b}|f'|^r\, d\rho d\theta} \leqslant B. \quad (9.2)$$

The inequalities are to be interpreted with the obvious conventions: if either integral is positive and finite, then the other is positive and finite and satisfies the inequalities. The condition $f(0) = 0$ is essential: the right-hand inequalities are obviously false, for example, when $f(z)$ is the constant 1.

The left-hand inequality (9.1) is a corollary of Theorem 3 and Lemma γ. Thus, by Theorem 3, with s for r and $c(\rho) = \mathfrak{M}_s(f, \rho)$,

$$\mathfrak{M}_s(f') \leqslant B\rho^{-\frac{1}{2}}(1-\rho)^{-1}\mathfrak{M}_s(f, \rho^{\frac{1}{2}}),$$

$$\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\, d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\rho^{-\frac{1}{2}r}\mathfrak{M}_s^r(f, \rho^{\frac{1}{2}})\, d\rho$$

$$= B\int_0^1 (1-\rho^2)^{-b}\rho^{1-r}\mathfrak{M}_s^r(f, \rho)\, d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\rho^{1-r}\mathfrak{M}_s^r(f, \rho)\, d\rho,$$

the last integral being convergent, at 1 by hypothesis, and at 0 because $c_0 = 0$ and so $\mathfrak{M}_s(f, \rho) = O(\rho)$. It now follows from Lemma γ that

$$\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\,d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho.$$

Passing to the right-hand inequality (9.1), we suppose first that $f(z)$ is regular for $\rho \leqslant 1$. Then, integrating by parts,*

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho = \frac{1}{1-b}\int_0^1 (1-\rho)^{1-b}\frac{d}{d\rho}\{\mathfrak{M}_s^r(f)\}\,d\rho.$$

But

$$\frac{d}{d\rho}\{\mathfrak{M}_s^r(f)\} = \frac{d}{d\rho}\{\mathfrak{M}_s^s(f)\}^{r/s} = \frac{r}{s}\{\mathfrak{M}_s^s(f)\}^{(r-s)/s}\frac{d}{d\rho}\{\mathfrak{M}_s^s(f)\}$$

$$= \frac{r}{s}\mathfrak{M}_s^{r-s}(f)\frac{d}{d\rho}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^s\,d\theta\right),$$

and

$$\left|\frac{d}{d\rho}|f|^s\right| \leqslant \left|\frac{d}{d\rho}f^s\right| = s|f|^{s-1}|f'|.$$

Hence

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho \leqslant B\int_0^1 (1-\rho)^{1-b}\mathfrak{M}_s^{r-s}(f)\,d\rho\int_{-\pi}^{\pi}|f|^{s-1}|f'|\,d\theta.$$

Also

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^{s-1}|f'|\,d\theta \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^s\,d\theta\right)^{(s-1)/s}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f'|^s\,d\theta\right)^{1/s}$$

$$= \mathfrak{M}_s^{s-1}(f)\mathfrak{M}_s(f'),$$

by Hölder's inequality; and hence

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho \leqslant B\int_0^1 (1-\rho)^{1-b}\mathfrak{M}_s^{r-1}(f)\mathfrak{M}_s(f')\,d\rho$$

$$= B\int_0^1 \{(1-\rho)^{-b}\mathfrak{M}_s^r(f)\}^{(r-1)/r}\{(1-\rho)^{r-b}\mathfrak{M}_s^r(f')\}^{1/r}\,d\rho$$

$$\leqslant B\left\{\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho\right\}^{(r-1)/r}\left\{\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\,d\rho\right\}^{1/r}$$

* And observing that $(1-\rho)^{1-b}\mathfrak{M}_s^r(f) = 0$ for $\rho = 0$ and $\rho = 1$.

(again by Hölder's inequality). Removing the common factor, and raising the result to the rth power, we obtain the right-hand inequality (9.1).

We have supposed $f(z)$ regular for $\rho \leqslant 1$. This condition is satisfied by $f(\sigma z)$, for any $\sigma < 1$. Hence, applying what we have proved to $f(\sigma z)$, we have

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r\{f(\sigma z)\}\,d\rho$$
$$\leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r\{\sigma f'(\sigma z)\}\,d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r\{f'(z)\}\,d\rho;$$

and so (using 'Fatou's Lemma')

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r\{f(z)\}\,d\rho$$
$$\leqslant \varliminf_{\sigma\to 1}\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r\{f(\sigma z)\}\,d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r\{f'(z)\}\,d\rho$$

in the general case.

The right-hand inequality (9.1) can be sharpened a little, and we shall need the refinement in the next section.

THEOREM 9. *Under the conditions of Theorem* 8

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\,d\rho. \tag{9.3}$$

For

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(f')\,d\rho$$
$$= B\int_0^1 (1-\rho)^{r-b}\rho^{-b}\mathfrak{M}_s^r(zf')\,d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\,d\rho,$$

the first inequality following from Theorem 8 and the last from Lemma γ.

10. Our next theorem is an extension of Theorem 8 to derivatives of non-integral order.

THEOREM 10. *If the conditions of Theorem* 8 *are satisfied, and* $0 < \beta < 1$, *then*

$$B \leqslant \frac{\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho}{\int_0^1 (1-\rho)^{\beta r-b}\mathfrak{M}_s^r(f^{\beta})\,d\rho} \leqslant B. \tag{10.1}$$

The left-hand inequality is a corollary of Theorem 7. For*

$$\mathfrak{M}_s^r(f^\beta) \leqslant \frac{B\mathfrak{M}_s^r(f, \rho^{\frac{1}{2}})}{(1-\rho)^{\beta r}},$$

by Theorem 7; and so

$$\int_0^1 (1-\rho)^{\beta r-b}\mathfrak{M}_s^r(f^\beta)\,d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f,\rho^{\frac{1}{2}})\,d\rho$$

$$= B\int_0^1 (1-\rho^2)^{-b}\rho\,\mathfrak{M}_s^r(f,\rho)\,d\rho \leqslant B\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho.$$

To prove the right-hand inequality we observe that

$$\int_0^1 (1-\rho)^{-b}\mathfrak{M}_s^r(f)\,d\rho \leqslant B\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\,d\rho, \tag{10.2}$$

by Theorem 9. But

$$zf' = c_1 z + 2c_2 z^2 + \ldots = -if^1$$

and

$$f^1 = (f^\beta)^{1-\beta}.$$

Hence, by the left-hand inequality, with f^β in place of f, and $1-\beta$ in place of β, we have

$$\int_0^1 (1-\rho)^{r-b}\mathfrak{M}_s^r(zf')\,d\rho \leqslant B\int_0^1 (1-\rho)^{\beta r-b}\mathfrak{M}_s^r(f^\beta)\,d\rho; \tag{10.3}$$

and the right-hand inequality follows from (10.2) and (10.3).

Another preliminary theorem

11. Theorem 11.† *If*

$$1 < r < s, \qquad \alpha = \frac{1}{r}-\frac{1}{s}, \qquad l \geqslant r, \tag{11.1}$$

and $\mathfrak{M}_r(f) \leqslant C$, *then*

$$\int_0^1 (1-\rho)^{l\alpha-1}\mathfrak{M}_s^l(f,\rho)\,d\rho \leqslant BC. \tag{11.2}$$

We may suppose $C = 1$.

* It will be observed that the proof is a little simpler than the corresponding part of the proof of Theorem 8. This is because f^β usually begins with a term in z, f' with a constant term. The result is true with $f^{(\beta)}$ for f^β, but the proof involves small complications like those of the proof of Theorem 8.

† Hardy and Littlewood, **6**, 411–14 (Theorem 31). There the theorem is proved for $0 < r < s$. A corollary is $\mathfrak{M}_s(f) = o\{(1-\rho)^{-\alpha}\}$.

It is sufficient to prove the theorem in the special case $l = r$. For

$$\mathfrak{M}_s(f) \leqslant B(1-\rho)^{-\alpha},$$

by Theorem 1, and so (if the special case has been proved)

$$\int_0^1 (1-\rho)^{l\alpha-1}\mathfrak{M}_s^l(f)\,d\rho = \int_0^1 (1-\rho)^{r\alpha-1}\mathfrak{M}_s^r(f)\{(1-\rho)^{\alpha}\mathfrak{M}_s(f)\}^{l-r}\,d\rho$$

$$\leqslant B\int_0^1 (1-\rho)^{r\alpha-1}\mathfrak{M}_s^r(f)\,d\rho \leqslant B.$$

We may therefore suppose that $l = r$.

Next, we may simplify the theorem as we simplified Theorem 1 in § 4. We may suppose first that f has no zeros in $\rho < 1$, and then (by putting $f = \phi^{2/r}$) that $r = 2$, in which case $s = q > 2$. We have then

$$r = 2,\ s > 2,\ \alpha = \frac{1}{2}-\frac{1}{s}, \qquad \sum |c_n|^2 = \mathfrak{M}_2^2(f) \leqslant 1;$$

and our conclusion is to be that

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(f)\,d\rho \leqslant B. \tag{11.3}$$

We write $$f = c_0+\sum_1^{\infty} c_n z^n = c_0+g.$$

Then $$\mathfrak{M}_q(f) \leqslant \mathfrak{M}_q(c_0)+\mathfrak{M}_q(g) = |c_0|+\mathfrak{M}_q(g)$$

and $$\mathfrak{M}_q^2(f) \leqslant 2|c_0|^2+2\mathfrak{M}_q^2(g). \tag{11.4}$$

Now $g(0) = 0$, and so

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(g)\,d\rho \leqslant B\int_0^1 (1-\rho)^{2-2/q}\mathfrak{M}_q^2(g')\,d\rho,$$

by Theorem 8. Also

$$\mathfrak{M}_q(g',\rho) \leqslant B(1-\rho)^{\frac{1}{q}-\frac{1}{2}}\mathfrak{M}_2(g',\rho) \leqslant B(1-\rho)^{\frac{1}{q}-\frac{1}{2}}\mathfrak{M}_2(g',\rho^{\frac{1}{2}}),$$

by Theorem 1. Hence

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(g)\,d\rho \leqslant B\int_0^1 (1-\rho)\mathfrak{M}_2^2(g',\rho^{\frac{1}{2}})\,d\rho$$

$$= B\int_0^1 (1-\rho)\Big(\sum_1^{\infty} n^2|c_n|^2\rho^{n-1}\Big)\,d\rho = B\sum_1^{\infty} n^2|c_n|^2\int_0^1 (1-\rho)\rho^{n-1}\,d\rho$$

$$= B\sum_1^{\infty}\frac{n}{n+1}|c_n|^2 \leqslant B\sum_1^{\infty}|c_n|^2 \leqslant B. \tag{11.5}$$

It then follows from (11.4) that

$$\int_0^1 (1-\rho)^{-2/q}\mathfrak{M}_q^2(f)\,d\rho \leqslant B|c_0|^2\frac{q}{q-2}+B \leqslant B,$$

which is (11.3); and this completes the proof.

The theorem '$r \to s$' and its extensions

12. The 'Hauptsatz' of our papers **3** and **6** was

THEOREM 12. *If* $$r < s, \qquad \alpha = \frac{1}{r}-\frac{1}{s}, \tag{12.1}$$

and $$f_\alpha(z) = \sum_1^\infty (in)^{-\alpha} c_n z^n, \tag{12.2}$$

then $$\mathfrak{M}_s(f_\alpha) \leqslant B\mathfrak{M}_r(f). \tag{12.3}$$

We proved this theorem there for $0 < r < s$; but now we are concerned only with the range $1 < r < s$.

Our proof of Theorem 12, for the range $1 < r < s$, was not at all 'function-theoretic', but was based on certain 'elementary' inequalities which we had proved in collaboration with Pólya.* Its logic was roughly as follows. We began by a rather difficult, though 'elementary', proof of the inequality

$$\left|\sum\sum{}' \frac{a_m b_n}{|m-n|^\lambda}\right| \leqslant B(\sum |a_m|^R)^{1/R}(\sum |b_n|^S)^{1/S}, \tag{12.4}$$

where

$$R > 1, \qquad S > 1, \qquad \frac{1}{R}+\frac{1}{S} > 1, \qquad \lambda = 2-\frac{1}{R}-\frac{1}{S},$$

and the dash excludes equal values of m and n from the summation. From this we passed, by standard limiting processes, to the integral inequality

$$\left|\int\int \frac{f(x)g(y)}{|x-y|^\lambda}\,dxdy\right| \leqslant B\Big(\int |f(x)|^R\,dx\Big)^{1/R}\Big(\int |g(y)|^S\,dy\Big)^{1/S}; \tag{12.5}$$

and from (12.5) we deduced, by the 'converse of Hölder's inequality', that

$$\Big(\int |f_\alpha(x)|^{S'}\,dx\Big)^{1/S'} \leqslant B\Big(\int |f(x)|^R\,dx\Big)^{1/R}, \tag{12.6}$$

where $$\frac{1}{S}+\frac{1}{S'} = 1, \qquad \alpha = \frac{1}{R}-\frac{1}{S'}, \tag{12.7}$$

* See Hardy, Littlewood, and Pólya, **8**, and **9**, ch. x; and Hardy and Littlewood, **3**, 568–76.

$f_\alpha(x)$, the integral of $f(x)$ of order α, is defined by

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int\limits_c^x (x-t)^{\alpha-1} f(t)\, db, \tag{12.8}$$

$0 < \alpha < 1$, and c may be $-\infty$ in certain circumstances.* Finally, we replaced R, S' by r, s and derived (12.3) from (12.6).†

The main theorems of our paper **7**, viz. Theorem 1 (which is proved) and Theorem 3 (which is stated without proof) are generalizations of Theorem 12. It seems desirable in any case to have a proof of Theorem 12 which is function-theoretic in character. Incidentally, we shall show (without going into details) how it would be possible to *deduce* (12.4) and (12.5) from (12.3). The only published proofs of these inequalities depend on difficult theorems concerning 'rearrangements'.‡

13. We begin by framing a generalization of Theorem 12. The function $f_\alpha(z)$ is substantially§ the Faltung of the two functions $f(z)$ and

$$g(z) = \sum n^{-\alpha} z^n; \tag{13.1}$$

and $g(z)$ satisfies

$$\mathfrak{M}_\sigma(g') \leqslant B(1-\rho)^{\alpha-1-1/\sigma'}, \tag{13.2}$$

where σ' is defined, as usual, by

$$\sigma' = \frac{\sigma}{\sigma-1}, \qquad \frac{1}{\sigma}+\frac{1}{\sigma'} = 1,$$

for any $\sigma \geqslant 1$.‖ It is therefore natural to replace the special function (13.1) by a general

$$g(z) = \sum_1^\infty b_n z^n \tag{13.3}$$

subject to (13.2) for some $\sigma \geqslant 1$.

* In particular when $f(x)$ has the period 2π and mean value 0 over a period. This is the case important in the theory of Fourier series.

† Afterwards extending it to the range $0 < r < s$ by 'function-theoretic' arguments. When $r \leqslant 1$ the result is essentially one about power-series, the analogue for general Fourier series (or harmonic functions) being false.

‡ See Hardy, Littlewood, and Pólya, **9**, ch. x, especially Theorems 371–3 and 379–82.

§ See § 15 for a precise statement.

‖ $1/\sigma' = 0$ when $\sigma = 1$. Since $|g'| \leqslant B|1-z|^{\alpha-2}$,

$$\mathfrak{M}_\sigma(g') \leqslant \left(B \int \frac{d\theta}{|1-z|^{\sigma(2-\alpha)}}\right)^{1/\sigma} \leqslant B(1-\rho)^{-2+\alpha+1/\sigma} = B(1-\rho)^{\alpha-1-1/\sigma'}.$$

We suppose then that
$$\mathfrak{M}_r(f) \leqslant 1 \tag{13.4}$$
and
$$\mathfrak{M}_\sigma(g') \leqslant (1-\rho)^{k-1}, \tag{13.5}$$
where
$$k = \alpha - \frac{1}{\sigma'}; \tag{13.6}$$
and write
$$h(z) = \sum_1^\infty b_n c_n z^n. \tag{13.7}$$
Analogy with Theorem 8 then suggests that
$$\mathfrak{M}_s(h) \leqslant B, \tag{13.8}$$
where
$$\frac{1}{s} = \frac{1}{r} - \alpha = \frac{1}{r} - k - \frac{1}{\sigma'}. \tag{13.9}$$
We must, however, impose some limitations on k. In the first place, we must suppose that
$$k < \frac{1}{r} - \frac{1}{\sigma'} \tag{13.10}$$
in order that s shall be positive. Secondly, as we shall see, we must suppose that $k \geqslant 0$. The cases $k > 0$ and $k = 0$ turn out to differ essentially, and we begin with the first.

14. Theorem 13. *If*
$$\sigma \geqslant 1, \qquad 0 < k < \frac{1}{r} - \frac{1}{\sigma'} \tag{14.1}$$
($1/\sigma'$ *being* 0 *if* $\sigma = 1$),
$$\frac{1}{s} = \frac{1}{r} - k - \frac{1}{\sigma'} \tag{14.2}$$
(*so that* $\quad 1 < r < s < \infty, \qquad r < \sigma', \qquad \sigma < s$);
$$f(z) = \sum_1^\infty c_n z^n, \qquad g(z) = \sum_1^\infty b_n z^n, \qquad h(z) = \sum_1^\infty b_n c_n z^n; \tag{14.3}$$
$$\mathfrak{M}_r(f) \leqslant 1, \qquad \mathfrak{M}_\sigma(g') \leqslant (1-\rho)^{k-1}; \tag{14.4}$$
then
$$\mathfrak{M}_s(h) \leqslant B. \tag{14.5}$$

It will be convenient to introduce a new parameter t defined by
$$\frac{1}{t} = \frac{1}{r} - k. \tag{14.6}$$
Then $r < t \leqslant s$ (and $t < s$ except when $\sigma = 1$).

We observe first that

$$h(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\{\rho^{\frac{1}{2}}e^{i(\theta-\phi)}\}g(\rho^{\frac{1}{2}}e^{i\phi})\,d\phi, \tag{14.7}$$

$$\rho e^{i\theta}h'(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\{\rho^{\frac{1}{2}}e^{i(\theta-\phi)}\}\rho^{\frac{1}{2}}e^{i\phi}g'(\rho^{\frac{1}{2}}e^{i\phi})\,d\phi; \tag{14.8}$$

and, if we take

$$\alpha = t, \qquad \beta = \sigma, \qquad \frac{1}{\gamma} = \frac{1}{t}+\frac{1}{\sigma}-1 = \frac{1}{r}-k-\frac{1}{\sigma'},$$

the conditions of Theorem A are satisfied. Hence

$$\begin{aligned}\rho^{\frac{1}{2}}\mathfrak{M}_s(h',\rho) &\leqslant \mathfrak{M}_t(f,\rho^{\frac{1}{2}})\mathfrak{M}_\sigma(g',\rho^{\frac{1}{2}})\\ &\leqslant (1-\rho^{\frac{1}{2}})^{k-1}\mathfrak{M}_t(f,\rho^{\frac{1}{2}}) \leqslant B(1-\rho)^{k-1}\mathfrak{M}_t(f,\rho^{\frac{1}{2}}).\end{aligned} \tag{14.9}$$

We must now distinguish three cases of the theorem, each needing a different proof. We suppose first that

$$1 < r < s \leqslant 2. \tag{14.10}$$

Case (a) of Theorem 13: $r < s \leqslant 2$

15. Since $s \leqslant 2$, we have

$$\mathfrak{M}_s^s(h) \leqslant B\int_0^1\int_{-\pi}^{\pi}(1-\rho)^{s-1}|h'|^s\,d\rho d\theta = B\int_0^1(1-\rho)^{s-1}\mathfrak{M}_s^s(h')\,d\rho, \tag{15.1}$$

by (2.13) of Theorem D. Hence, using (14.9) and Lemma γ, we obtain

$$\begin{aligned}\mathfrak{M}_s^s(h) &\leqslant B\int_0^1(1-\rho)^{sk-1}\rho^{-\frac{1}{2}s}\mathfrak{M}_t^s(f,\rho^{\frac{1}{2}})\,d\rho\\ &= B\int_0^1(1-\rho^2)^{sk-1}\rho^{1-s}\mathfrak{M}_t^s(f,\rho)\,d\rho \leqslant B\int_0^1(1-\rho)^{sk-1}\rho^{1-s}\mathfrak{M}_t^s(f)\,d\rho\\ &\leqslant B\int_0^1(1-\rho)^{sk-1}\mathfrak{M}_t^s(f)\,d\rho.\end{aligned} \tag{15.2}$$

Finally, $s > r$ and $t > r$, and therefore

$$\int_0^1(1-\rho)^{sk-1}\mathfrak{M}_t^s(f)\,d\rho \leqslant B,$$

by Theorem 11.

Case (b): $r \leqslant 2 \leqslant s$

16. We suppose next that $r \leqslant 2 \leqslant s$. Since (15.1) may now be false, we use Theorem E instead of Theorem D. We have

$$\mathfrak{M}_s^s(h) \leqslant B \int_{-\pi}^{\pi} d\theta \left\{ \int_0^1 (1-\rho)|h'|^2 \, d\rho \right\}^{\frac{1}{2}s}.$$

But

$$\left[\int_{-\pi}^{\pi} d\theta \left\{ \int_0^1 (1-\rho)|h'|^2 \, d\rho \right\}^{\frac{1}{2}s} \right]^{2/s} \leqslant \int_0^1 (1-\rho)\, d\rho \left(\int_{-\pi}^{\pi} |h'|^s \, d\theta \right)^{2/s},$$

by Minkowski's inequality.* Hence

$$\mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)\mathfrak{M}_s^2(h')\, d\rho \leqslant B \int_0^1 (1-\rho)^{2k-1}\rho^{-1}\mathfrak{M}_t^2(f, \rho^{\frac{1}{2}})\, d\rho$$

$$= B \int_0^1 (1-\rho^2)^{2k-1}\rho^{-1}\mathfrak{M}_t^2(f,\rho)\, d\rho \leqslant B \int_0^1 (1-\rho)^{2k-1}\rho^{-1}\mathfrak{M}_t^2(f,\rho)\, d\rho$$

$$\leqslant B \int_0^1 (1-\rho)^{2k-1}\mathfrak{M}_t^2(f,\rho)\, d\rho,$$

by (14.9) and Lemma γ. The conclusion now follows from Theorem 11, since $2 \geqslant r$ and $t > r$.

Case (c): $2 < r < s$

17. When finally $2 < r < s$, we have to use a quite different method. Actually, we deduce Case (*c*) from Case (*a*) by a 'conjugacy' argument, here of a very simple type.†

Suppose that
$$\psi(\theta) = \sum_{-N}^{N} \gamma_n e^{ni\theta} \tag{17.1}$$

is an arbitrary trigonometrical polynomial. It will be convenient to write

$$\psi = \sum_{-N}^{N} = \sum_{1}^{N} + \sum_{-N}^{0} = \psi_1 + \psi_2. \tag{17.2}$$

Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h(\rho^2 e^{-i\theta})\psi(\theta)\, d\theta = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \psi(\theta)\, d\theta \int_{-\pi}^{\pi} f(\rho e^{-i\phi}) g(\rho e^{i\phi - i\theta})\, d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{-i\phi}) H(\rho e^{i\phi})\, d\phi, \tag{17.3}$$

* In the form used in the proof of Lemma α (§ 3).

† There is a more difficult specimen in Littlewood and Paley, **11**, § 8.

where

$$H(\rho e^{i\phi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(\theta) g(\rho e^{i\phi - i\theta})\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_1(\theta) g(\rho e^{i\phi - i\theta})\, d\theta = \sum_1^N b_n \gamma_n \rho^n e^{ni\phi}. \qquad (17.4)$$

If we write for the moment

$$\mathfrak{f}(z) = \sum_1^N \gamma_n z^n, \qquad \mathfrak{g}(z) = g(z), \qquad \mathfrak{h}(z) = H(z),$$

so that

$$\mathfrak{f}(e^{i\theta}) = \psi_1(\theta),$$

then $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$ are related as f, g, and h are related in the main theorem. Also, if

$$\frac{1}{r}+\frac{1}{r'} = \frac{1}{s}+\frac{1}{s'} = 1,$$

then

$$1 < s' < r' < 2$$

and

$$\frac{1}{s'}-\frac{1}{r'} = \frac{1}{r}-\frac{1}{s}.$$

We may therefore apply Case (a) of the theorem to $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, with s', r' for r, s; and this gives

$$\mathfrak{M}_{r'}\{H(\rho e^{i\theta})\} \leqslant B\mathfrak{M}_{s'}\{\psi_1(\theta)\}.$$

But

$$\mathfrak{M}_{s'}\{\psi_1(\theta)\} \leqslant B\mathfrak{M}_{s'}\{\psi(\theta)\},$$

by Theorem F;* and so

$$\mathfrak{M}_{r'}\{H(\rho e^{i\theta})\} \leqslant B\mathfrak{M}_{s'}\{\psi(\theta)\}. \qquad (17.5)$$

Also

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{-i\phi}) H(\rho e^{i\phi})\, d\phi\right| \leqslant \mathfrak{M}_r(f)\mathfrak{M}_{r'}(H) \leqslant \mathfrak{M}_{r'}(H), \qquad (17.6)$$

by Hölder's inequality and (14.4). Hence, collecting our results from (17.3), (17.6), and (17.5), we obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\rho^2 e^{-i\theta})\psi(\theta)\, d\theta\right| \leqslant B\mathfrak{M}_{s'}\{\psi(\theta)\}. \qquad (17.7)$$

* In the form (2.22).

This is true for all trigonometrical polynomials ψ, and so for all ψ of $L^{s'}$; and therefore, by the converse of Hölder's inequality,*

$$\mathfrak{M}_s\{h(\rho^2 e^{i\theta})\} = \mathfrak{M}_s\{h(\rho^2 e^{-i\theta})\} \leqslant B.$$

This completes the proof of Theorem 13.†

In particular, if we take $g(z) = \sum n^{-\alpha} z^n$, we obtain Theorem 12.

18. We return for a moment to our remarks about Theorem 12 in § 12. We have now a proof of this theorem independent of the inequality (12.4) and of the theorems on 'rearrangements' on which that inequality was based. It is interesting to observe (without attempting to carry out the process in detail) how we could reverse the old argument and *deduce* (12.4) from Theorem 12.

We should begin by passing from

$$f_\alpha = \sum (in)^{-\alpha} c_n z^n$$

to the fractional integral of an arbitrary real function $f(x)$, defined as in (12.8). Here c might be finite (as with Riemann and Liouville) or $-\infty$ (as with Weyl). The deduction would involve only arguments similar to those of § 8, and processes of approximation of standard types. We should thus prove (12.6), and from this deduce that

$$\left|\iint |x-y|^{\alpha-1} f(x) g(y)\, dxdy\right| \leqslant B\left(\int |f(x)|^r\, dx\right)^{1/r}\left(\int |g(y)|^{s'}\, dy\right)^{1/s'}.$$

Since $$1-\alpha = 1-\frac{1}{r}+\frac{1}{s} = 2-\frac{1}{r}-\frac{1}{s'},$$

this is equivalent to (12.5); and the passage back to (12.4) is straightforward.

* 'If $k > 1$ and $\quad \left|\int FG\, dx\right| \leqslant C\left(\int |G|^{k'}\, dx\right)^{1/k'}$

for all G of $L^{k'}$, then $\quad \left(\int |F|^k\, dx\right)^{1/k} \leqslant C.$'

See, for example, Hardy, Littlewood, and Pólya, **9**, 142 (Theorem 191): the theorem was first proved by F. Riesz. It remains true if G is confined to one of the standard classes of approximating functions (step-functions, polynomials, or trigonometrical polynomials).

† There is an analogue of Theorem 13 for harmonic functions which an experienced reader will be able to state and prove for himself. When $\sigma > 1$, the proof depends on Theorem F (and Theorem 13 itself). When $\sigma = 1$ we require an additional weapon: *if* $k < 1$ *and*

$$\mathfrak{M}_1(u) \leqslant (1-\rho)^{k-1},$$

then $$\mathfrak{M}_1(u_1) \leqslant B(1-\rho)^{k-1}, \qquad \mathfrak{M}_1(u_2) \leqslant B(1-\rho)^{k-1}.$$

The case $k = 0$

19. In Theorem 13 we supposed that $k > 0$, and this assumption was essential to the proof. Thus if $k = 0$, $t = r$, and the appeal to Theorem 11, at the end of § 15, is no longer justified. Actually, as we shall see, the theorem becomes false when $k = 0$ and $r < s < 2$ or $2 < r < s$, i.e. in cases (*a*) and (*c*).

Case (*b*) of the theorem, however, survives, as we proved in **7**. The theorem which follows is in fact Theorem 1 of **7**, but the proof which we give here is shorter.*

THEOREM 14. *If f, g, and h are defined as in Theorem* 13;

$$1 \leqslant r \leqslant 2 \leqslant s; \tag{19.1}$$

$$\sigma \geqslant 1, \qquad r < \sigma', \qquad \frac{1}{s} = \frac{1}{r} - \frac{1}{\sigma'}; \tag{19.2}$$

$$\mathfrak{M}_r(f) \leqslant 1, \qquad \mathfrak{M}_\sigma(g') \leqslant \frac{1}{1-\rho}; \tag{19.3}$$

then

$$\mathfrak{M}_s(h) \leqslant B. \tag{19.4}$$

Since we shall be concerned (in the statement and proof of the theorem) only with power-series, we have departed from our practice in the rest of the paper by including the case $r = 1$,† which is in fact particularly interesting.‡

20. We begin with a trivial simplification of the theorem, showing that (as the reader will readily believe) it is sufficient to prove it in the special case in which $c_1 = 0$ and $b_1 = 0$.

Suppose that it has been proved in this case, and let

$$F(z) = f(z) - c_1 z, \qquad G(z) = g(z) - b_1 z, \qquad H(z) = h(z) - b_1 c_1 z.$$

Then

$$\mathfrak{M}_r(F) \leqslant \mathfrak{M}_r(f) + |c_1|\rho \leqslant \mathfrak{M}_r(f) + \mathfrak{M}_1(f) \leqslant 2\mathfrak{M}_r(f) \leqslant 2$$

and

$$\mathfrak{M}_\sigma(G') \leqslant \mathfrak{M}_\sigma(g') + |b_1| \leqslant \mathfrak{M}_\sigma(g') + \mathfrak{M}_1(g') \leqslant 2\mathfrak{M}_\sigma(g') \leqslant \frac{2}{1-\rho}.$$

Hence, by our assumption,

$$\mathfrak{M}_s(H) \leqslant 4B = B,$$

* Though, as we stated in § 1, it depends on more difficult theorems.

† We used harmonic polynomials in the proof of case (*c*) of Theorem 13 (§ 17). Theorem 14, like Theorem 13, can be extended to harmonic functions when $r > 1$, but not when $r = 1$. See **7**, 162, footnote †.

‡ It includes the theorem of Paley referred to in **7**.

and therefore

$$\mathfrak{M}_s(h) \leqslant \mathfrak{M}_s(H) + |b_1||c_1|\rho \leqslant B + \mathfrak{M}_r(f)\mathfrak{M}_\sigma(g', 0) \leqslant B+1 = B;$$

so that the theorem is true generally.

We suppose then in what follows that $b_1 = 0$ and $c_1 = 0$. By Theorem E, we have

$$\lim_{\rho\to 1} \mathfrak{M}_s^s(h) \leqslant B \int_{-\pi}^{\pi} d\theta \left\{ \int_0^1 (1-\rho)|h'|^2 \, d\rho \right\}^{\frac{1}{2}s};$$

and (using Minkowski's inequality as in § 16) we deduce

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)\mathfrak{M}_s^2(h')\, d\rho. \tag{20.1}$$

If $0 < \tau < 1$ and

$$k(\zeta) = k(\tau e^{i\theta}) = h'(\rho\tau e^{i\theta}),$$

then $k(0) = 0$,* and we may apply (20.1) to $k(\zeta)$. We have thus

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(k) \leqslant B \int_0^1 (1-\tau)\mathfrak{M}_s^2(k', \tau)\, d\tau,$$

i.e.

$$\mathfrak{M}_s^2(h', \rho) \leqslant B \int_0^1 (1-\tau)\mathfrak{M}_s^2\{\rho h''(\rho\tau e^{i\theta})\}\, d\tau$$

$$= B\rho^2 \int_0^1 (1-\tau)\mathfrak{M}_s^2\{h''(\rho\tau e^{i\theta})\}\, d\tau = B \int_0^\rho (\rho-\omega)\mathfrak{M}_s^2\{h''(\omega e^{i\theta})\}\, d\omega.$$

Substituting in (20.1), we obtain

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)\, d\rho \int_0^\rho (\rho-\omega)\mathfrak{M}_s^2(h'', \omega)\, d\omega$$

$$= B \int_0^1 \mathfrak{M}_s^2(h'', \omega)\, d\omega \int_\omega^1 (1-\rho)(\rho-\omega)\, d\rho;$$

and so, performing the inner integration and then replacing ω by ρ,

$$\lim_{\rho\to 1} \mathfrak{M}_s^2(h) \leqslant B \int_0^1 (1-\rho)^3\mathfrak{M}_s^2(h'', \rho)\, d\rho. \tag{20.2}$$

If ϑ is the operator $\qquad \vartheta = z\dfrac{d}{dz}$,

* Here we use $c_1 = 0$ (or $b_1 = 0$).

then

$$\vartheta f = \sum nc_n z^n, \qquad \vartheta g = \sum nb_n z^n, \qquad \vartheta^2 h = \sum n^2 b_n c_n z^n,$$

and
$$\vartheta^2 h(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \vartheta f(\rho^{\frac{1}{2}}e^{i\phi})\, \vartheta g(\rho^{\frac{1}{2}}e^{i\theta-i\phi})\, d\phi.$$

Hence, by Theorem A,

$$\mathfrak{M}_s(\vartheta^2 h, \rho) \leqslant \mathfrak{M}_r(\vartheta f, \rho^{\frac{1}{2}})\mathfrak{M}_\sigma(\vartheta g, \rho^{\frac{1}{2}}). \tag{20.3}$$

Also
$$\vartheta h(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho^{\frac{1}{2}}e^{i\phi})\, \vartheta g(\rho^{\frac{1}{2}}e^{i\theta-i\phi})\, d\phi$$

and
$$\mathfrak{M}_s(\vartheta h, \rho) \leqslant \mathfrak{M}_r(f, \rho^{\frac{1}{2}})\mathfrak{M}_\sigma(\vartheta g, \rho^{\frac{1}{2}}). \tag{20.4}$$

Now
$$\vartheta^2 h = z^2 h'' + zh, \qquad h'' = z^{-2}\vartheta^2 h - z^{-2}\vartheta h.$$

Hence
$$\mathfrak{M}_s^2(h'') \leqslant B\{\rho^{-4}\mathfrak{M}_s^2(\vartheta^2 h) + \rho^{-4}\mathfrak{M}_s^2(\vartheta h)\}$$

and, by (20.2),

$$\mathfrak{M}_s^2(h) \leqslant \lim_{\rho\to 1} \mathfrak{M}_s^2(h) \leqslant B(J_1 + J_2), \tag{20.5}$$

where
$$J_1 = \int_0^1 (1-\rho)^3 \rho^{-4}\mathfrak{M}_s^2(\vartheta^2 h)\, d\rho, \tag{20.6}$$

and
$$J_2 = \int_0^1 (1-\rho)^3 \rho^{-4}\mathfrak{M}_s^2(\vartheta h)\, d\rho. \tag{20.7}$$

The theorem will therefore be proved if we can show that $J_1 \leqslant B$ and $J_2 \leqslant B$.

21. Since $\vartheta^2 h = O(\rho^2)$ for small ρ, J_1 is convergent at the origin, and therefore

$$\begin{aligned} J_1 \leqslant B\int_0^1 (1-\rho)^3\mathfrak{M}_s^2(\vartheta^2 h)\, d\rho &\leqslant B\int_0^1 (1-\rho)^3\mathfrak{M}_r^2(\vartheta f, \rho^{\frac{1}{2}})\mathfrak{M}_\sigma^2(\vartheta g, \rho^{\frac{1}{2}})\, d\rho \\ &= B\int_0^1 \rho(1-\rho^2)^3\mathfrak{M}_r^2(\vartheta f, \rho)\mathfrak{M}_\sigma^2(\vartheta g, \rho)\, d\rho \\ &\leqslant B\int_0^1 (1-\rho)\mathfrak{M}_r^2(f', \rho)\, d\rho, \end{aligned} \tag{21.1}$$

by Lemma γ, (20.3) and (19.3). But

$$\left\{\int_0^1 (1-\rho)\mathfrak{M}_r^2(f')\,d\rho\right\}^{\frac{1}{2}r} = \left\{\int_0^1 (1-\rho)\,d\rho\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'|^r\,d\theta\right)^{2/r}\right\}^{\frac{1}{2}r}$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\left\{\int_0^1 (1-\rho)|f'|^2\,d\theta\right\}^{\frac{1}{2}r} \leqslant B\mathfrak{M}_r(f) \leqslant B, \quad (21.2)$$

by Minkowski's inequality and Theorem E. From (21.1) and (21.2) it follows that $J_1 \leqslant B$.

There remains J_2.* Here we have

$$J_2 \leqslant B\int_0^1 (1-\rho)^3\mathfrak{M}_s^2(\vartheta h)\,d\rho \leqslant B\int_0^1 (1-\rho)^3\mathfrak{M}_r^2(f,\rho^{\frac{1}{2}})\mathfrak{M}_\sigma^2(\vartheta g,\rho^{\frac{1}{2}})\,d\rho$$

$$\leqslant B\int_0^1 (1-\rho)^3(1-\rho^{\frac{1}{2}})^{-2}\,d\rho = B,$$

by Lemma γ, (20.4), and (19.3); and this completes the proof.

The limitations of the theorems

22. The rest of the paper is negative: we construct examples to show that the restrictions which we have imposed on k, r, and s are essential for the truth of our theorems. We have supposed

(i) that $k \geqslant 0$,

(ii) that $2 \leqslant r \leqslant s$ when $k = 0$.

We shall now show

(1) that all our conclusions would be false were $k < 0$,

(2) that the conclusion of Theorem 14 (in which $k = 0$) would be false were $r \leqslant s < 2$ or $2 < r \leqslant s$.

23. To prove (1), we take

$$f(z) = \sum n^{-2}z^{2^n}, \quad g(z) = \sum 2^{-kn}z^{2^n}, \quad h(z) = \sum n^{-2}2^{-kn}z^{2^n} \quad (23.1)$$

(the summations being from 0 to ∞). Then $f(z)$ is continuous in $\rho \leqslant 1$, and

$$\mathfrak{M}_r(f) \leqslant A \quad (23.2)$$

for every r.

Next, we write $a = 1-k$ and $\rho = e^{-\delta}$. Then

$$zg'(z) = \sum 2^{an}z^{2^n}.$$

* Which is really trivial.

Now $g'(z)$ is bounded in $\rho \leqslant e^{-a}$, so that

$$|g'(z)| \leqslant B \leqslant B(1-\rho)^{-a} = B(1-\rho)^{k-1} \tag{23.3}$$

if $\rho \leqslant e^{-a}$, i.e. if $\delta \geqslant a$. On the other hand, if $\delta < a$, the function

$$\phi(x) = 2^{ax}e^{-2^x\delta},$$

where $x \geqslant 0$, has a single maximum, equal to

$$\left(\frac{a}{\delta}\right)^a e^{-a},$$

when $$x = \xi, \qquad 2^\xi = a/\delta.$$

Also $$|zg'(z)| \leqslant \sum 2^{an}e^{-2^n\delta} = \sum \phi(n),$$

and

$$\sum \phi(n) \leqslant 2\phi(\xi) + \int\limits_0^\infty \phi(x)\,dx = 2\left(\frac{a}{\delta}\right)^a e^{-a} + \int\limits_0^\infty 2^{ax}e^{-2^x\delta}\,dx$$

$$= 2\left(\frac{a}{\delta}\right)^a e^{-a} + A\int\limits_1^\infty y^{a-1}e^{-\delta y}\,dy < \{2a^a e^{-a} + A\Gamma(a)\}\delta^{-a}$$

$$= B\delta^{-a} \leqslant B(1-\rho)^{-a} = B(1-\rho)^{k-1};$$

so that $$|g'(z)| < Be^a(1-\rho)^{k-1} = B(1-\rho)^{k-1} \tag{23.4}$$

if $\delta < a$.

It follows from (23.3) and (23.4) that

$$\mathfrak{M}_\sigma(g') \leqslant \mathfrak{M}(g') \leqslant B(1-\rho)^{k-1}$$

for $0 \leqslant \rho < 1$ and any σ. Thus f and g satisfy the requirements of Theorem 13. But the coefficients of h do not tend to 0, and so $\mathfrak{M}_s(h)$ cannot be bounded for any $s \geqslant 1$.

24. It is more difficult to prove assertion (2) of § 22, and we do not attempt so comprehensive a refutation of any extension of Theorem 14. We examine a particular, and representative case: even there, we cannot produce a definite 'Gegenbeispiel', though we can show that such exist.

The two cases, corresponding to cases (a) and (c) of Theorem 13, would stand or fall together (by the 'conjugacy' argument used in § 17). We may therefore suppose that $2 < r \leqslant s$, and we select the case

$$\sigma = 1, \qquad \sigma' = \infty, \qquad r = 4, \qquad s = 4.^* \tag{24.1}$$

* $r = s$ if $k = 0$ and $\sigma = 1$. In Theorem 13, s is necessarily greater than r; and this is true in Theorem 14 also except when $r = s = 2$. But here the case $r = s$ is representative.

We shall prove that

$$\mathfrak{M}_4(f) \leqslant 1, \qquad \mathfrak{M}_1(g') \leqslant \frac{1}{1-\rho} \tag{24.2}$$

do not imply

$$\mathfrak{M}_4(h) \leqslant A. \tag{24.3}$$

It will plainly be enough to show that, given any Ω, we can find two A's and an f and g such that

$$\mathfrak{M}_4(f) \leqslant A, \qquad \mathfrak{M}_1(g') \leqslant A, \tag{24.4}$$

and

$$\mathfrak{M}_4(h) > \Omega. \tag{24.5}$$

We write

$$D_n(z) = D_n(\rho e^{i\theta}) = \sum_{N_n}^{N_n+n-1} z^p = z^{N_n}\frac{1-z^n}{1-z}, \tag{24.6}$$

where

$$N = 2^{2^n}. \tag{24.7}$$

There is plainly no overlapping of the powers of z in different D_n. We then define f and g by

$$f(z) = \sum_2^{\nu} n^{-\beta} D_n(ze^{i\alpha_n}), \tag{24.8}$$

$$g(z) = \sum_2^{\infty} \frac{1}{\log n} D_n(ze^{-i\alpha_n}); \tag{24.9}$$

so that

$$h(z) = \sum_2^{\nu} \frac{n^{-\beta}}{\log n} D_n(z). \tag{24.10}$$

Here

$$\beta = \tfrac{9}{8}, \tag{24.11}$$

the α_n are arbitrary, and ν is large; f and h are polynomials, and h is independent of the choice of the α_n.

25.* We prove first that

$$\mathfrak{M}_1(g') \leqslant \frac{A}{1-\rho}. \tag{25.1}$$

In proving this we may suppose that $\rho \geqslant \frac{15}{16}$.

Let

$$\rho_n = 1 - \frac{1}{N_n}, \tag{25.2}$$

and suppose that

$$\rho_{m-1} \leqslant \rho < \rho_m, \tag{25.3}$$

* The only part of the argument of this section which is not really trivial is the proof of (25.12). It is there only that we use the $\log n$ in (24.10), or the distinction between $\mathfrak{M}_1$ and $\mathfrak{M}$.

where $m > 2$.* Then

$$g'(z) = \sum_{2}^{\infty} \frac{1}{\log n} \frac{d}{dz} D_n(ze^{-i\alpha_n}), \tag{25.4}$$

$$|g'(z)| \leqslant \sum_{2}^{\infty} \frac{1}{\log n} |D'_n(ze^{-i\alpha_n})| = \gamma_1(z)+\gamma_2(z)+\gamma_3(z), \tag{25.5}$$

where γ_1 contains the terms for which $n < m-1$ (if any), γ_2 those for which $n = m-1$ and $n = m$, and γ_3 those for which $n > m$.

It is plain that

$$|D'_n(ze^{-i\alpha_n})| \leqslant \sum_{p=0}^{n-1} (N_n+p)\rho^{N_n+p-1} \leqslant 2\rho^{N_n} \sum_{p=0}^{n-1} (N_n+p) \leqslant 4nN_n \rho^{N_n} < 4nN_n. \tag{25.6}$$

Hence, first

$$\gamma_1(z) \leqslant 4 \sum_{2}^{m-2} nN_n \leqslant 4m(m-2)2^{2^{m-2}} < 4.2^{2^{m-1}} = 4N_{m-1} = \frac{4}{1-\rho_{m-1}} \leqslant \frac{4}{1-\rho}.\dagger \tag{25.7}$$

Secondly

$$\gamma_3(z) \leqslant 4 \sum_{m+1}^{\infty} nN_n \rho^{N_n} < 4 \sum_{m+1}^{\infty} nN_n \left(1-\frac{1}{N_m}\right)^{N_n} < 4 \sum_{m+1}^{\infty} nN_n e^{-N_n/N_m}$$

$$= 4 \sum_{m+1}^{\infty} n2^{2^n} \exp(-2^{2^n-2^m}) < 4 \sum_{m+1}^{\infty} n2^{2^n} \exp(-2^{2^{n-1}}) < 4 \sum_{1}^{\infty} n2^{-2^n} \ddagger$$

$$= 4(2^{-2}+2.2^{-4}+3.2^{-8}+...) < 4 < \frac{4}{1-\rho}. \tag{25.8}$$

From (25.7) and (25.8) it follows that

$$\mathfrak{M}_1(\gamma_1)+\mathfrak{M}_1(\gamma_3) \leqslant \mathfrak{M}(\gamma_1)+\mathfrak{M}(\gamma_3) < \frac{8}{1-\rho}. \tag{25.9}$$

As regards γ_2, we have

$$D'_n(ze^{-i\alpha_n}) = D'_n(\zeta) = N_n \zeta^{N_n-1} \frac{1-\zeta^n}{1-\zeta} + \zeta^{N_n} \frac{d}{d\zeta}\left(\frac{1-\zeta^n}{1-\zeta}\right) = P_n+Q_n,$$

* So that $\rho \geqslant \rho_2 = 1-\frac{1}{N_2} = \frac{15}{16}$.

† $m(m-2) < 2^{2^{m-2}}$ for $m \geqslant 2$.

‡ $e > 2$, $2^{2^n} > 2.2^{2^{n-1}}$, and $2^{2^{n-1}} > 2^{n+1}$, for $n > 3$.

say; and

$$|\gamma_2(z)| \leqslant \sum_{m-1}^{m} \frac{|P_n|}{\log n} + \sum_{m-1}^{m} \frac{|Q_n|}{\log n} = \lambda(z)+\mu(z), \tag{25.10}$$

say. Now

$$|Q_n| < 1+2+...+(n-1) < n^2 \leqslant m^2 < N_{m-1} = \frac{1}{1-\rho_{m-1}} \leqslant \frac{1}{1-\rho},$$

and so

$$\mathfrak{M}_1(\mu) \leqslant \mathfrak{M}(\mu) < \frac{4}{1-\rho} \tag{25.11}$$

(since $\log 2 > \frac{1}{2}$). Also

$$\mathfrak{M}_1(P_n) = N_n \rho^{N_n-1} \int_{-\pi}^{\pi} \left|\frac{1-\zeta^n}{1-\zeta}\right| d\theta \leqslant AN_n \rho^{N_n} \log n,$$

by Lemma δ. The maximum of $x\rho^x$, for fixed $\rho < 1$ and positive x, is

$$\frac{A}{\log(1/\rho)} < \frac{A}{1-\rho};$$

and so (whether n be $m-1$ or m)

$$\frac{\mathfrak{M}_1(P_n)}{\log n} < \frac{1}{\log n} . A \log n . \frac{A}{1-\rho} = \frac{A}{1-\rho}.$$

Hence

$$\mathfrak{M}_1(\lambda) < \frac{A}{1-\rho}. \tag{25.12}$$

Finally, (25.1) follows from (25.5), (25.9), (25.10), (25.11), and (25.12).

26. Next, we prove that

$$\mathfrak{M}_4(h) > \Omega \tag{26.1}$$

if $z = e^{i\theta}$ and ν is sufficiently large. We write

$$\phi_{m,n}(z) = \frac{D_m(z)}{m^\beta \log m} \frac{D_n(z)}{n^\beta \log n} = \sum_k c_{m,n,k} z^k:$$

$\phi_{m,n}$ is a polynomial with positive coefficients. Then

$$h^2(z) = \sum_{m,n} \phi_{m,n}(z) = \sum_k c_k z^k,$$

where

$$c_k = \sum_{m,n} c_{m,n,k};$$

and

$$\mathfrak{M}_4^4(h) = \sum_k c_k^2.$$

Since

$$\sum_k c_k^2 = \sum_k \Big(\sum_{m,n} c_{m,n,k}\Big)^2 > \sum_{m,n,k} c_{m,n,k}^2 = \sum_{m,n} \mathfrak{M}_2^2(\phi_{m,n}),$$

we have

$$\mathfrak{M}_4^4(h) \geqslant \sum_{m,n} \left(\frac{1}{m^\beta \log m}\right)^2 \left(\frac{1}{n^\beta \log n}\right)^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(z)D_n(z)|^2 \, d\theta. \quad (26.2)$$

It will be sufficient to consider the sum over the range

$$1 < \tfrac{1}{2}n \leqslant m \leqslant n.$$

Since

$$D_m(z)D_n(z) = z^{N_m+N_n}(1+z+...+z^{m-1})(1+z+...+z^{n-1}),$$

and the product of the last two factors contains the terms

$$1+2z+3z^2+...+mz^{m-1},$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(z)D_n(z)|^2 \, d\theta \geqslant 1^2+2^2+...+m^2 > \tfrac{1}{3}m^3.$$

Substituting in (26.2), we see that

$$\mathfrak{M}_4^4(h) \geqslant \frac{1}{3} \sum_{n=2}^{\nu} \frac{1}{n^{2\beta}(\log n)^2} \sum_{m=\frac{1}{2}n+1}^{n} \frac{m^3}{m^{2\beta}(\log m)^2}$$

$$\geqslant A \sum_{2}^{\nu} \frac{n^{4-4\beta}}{(\log n)^4} = A \sum_{2}^{\nu} \frac{n^{-1}}{(\log n)^4} > \Omega^4,$$

if ν is sufficiently large.

27. So far our results have been true however the α_n are chosen. We shall now prove that

$$\mathfrak{M}_4\{f(e^{i\theta})\} \leqslant A \quad (27.1)$$

for *some* choice of the α_n. We denote by $\mathfrak{A}_\alpha$ the operator

$$\left(\frac{1}{2\pi}\right)^{\nu-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} ... d\alpha_2 \, d\alpha_3 ... d\alpha_\nu$$

(an average over all different values of the α_n). It is plain that, if

$$\mathfrak{A}_\alpha[\mathfrak{M}_4^4\{f(e^{i\theta})\}] \leqslant A^4, \quad (27.2)$$

then (27.1) must be true for some set of α_n.

We shall now prove (27.2). We shall suppose throughout that z lies on the unit circle (as we may, since f is a polynomial); and f will mean $f(e^{i\theta})$.

We write $\quad D_n(ze^{i\alpha_n}) = D_n\{e^{i(\theta+\alpha_n)}\} = \Delta_n(\theta+\alpha_n), \quad (27.3)$

and $\bar{\Delta}_n$ for the conjugate of Δ_n. Then

$$\mathfrak{M}_4^4(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^4\,d\theta$$

$$= \sum_{m,n,p,q} (mnpq)^{-\beta}\frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_m(\theta+\alpha_m)\Delta_n(\theta+\alpha_n)\bar{\Delta}_p(\theta+\alpha_p)\bar{\Delta}_q(\theta+\alpha_q)\,d\theta. \tag{27.4}$$

Since

$$\Delta_m(\theta+\alpha_m) = e^{iN_m(\theta+\alpha_m)}\{1+e^{i(\theta+\alpha_m)}+\ldots+e^{i(m-1)(\theta+\alpha_m)}\},$$

$$\bar{\Delta}_p(\theta+\alpha_p) = e^{-iN_p(\theta+\alpha_p)}\{1+e^{-i(\theta+\alpha_p)}+\ldots+e^{-i(p-1)(\theta+\alpha_p)}\},$$

integration over the α destroys all the terms in (27.4) except those for which either $p = m$ and $q = n$, or else $p = n$ and $q = m$: consequently

$$\mathfrak{A}_\alpha\{\mathfrak{M}_4^4(f)\}$$

$$\leqslant 2\sum_{m,n}(mn)^{-2\beta}\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2|\Delta_n(\theta+\alpha_n)|^2\,d\theta\right\} = 2(S_1+S_2), \tag{27.5}$$

where S_1 contains the terms for which $m = n$ and S_2 the remainder.

First, $$S_1 = \sum n^{-4\beta}\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)^4|\,d\theta\right\}.$$

But

$$\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^4\,d\theta\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha_n\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^4\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\psi)|^4\,d\psi$$

(since the inner integral has this value for every value of α_n). Also

$$|\Delta_n(\psi)| \leqslant n;$$

and so $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\psi)|^4\,d\psi \leqslant \frac{n^2}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\psi)|^2\,d\psi = n^3.$$

Hence $$\mathfrak{A}_\alpha\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^4\,d\theta\right\} \leqslant n^3,$$

and $$S_1 \leqslant \sum n^{3-4\beta} = \sum n^{-\frac{4}{3}} = A. \tag{27.6}$$

On the other hand

$$S_2 = \sum_{m \neq n} (mn)^{-2\beta} \mathfrak{A}_\alpha \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 |\Delta_n(\theta+\alpha_n)|^2 \, d\theta \right\}.$$

But

$$\mathfrak{A}_\alpha \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 |\Delta_n(\theta+\alpha_n)|^2 \, d\theta \right\}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha_m \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha_n \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 |\Delta_n(\theta+\alpha_n)|^2 \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\theta+\alpha_m)|^2 \, d\alpha_m \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_n(\theta+\alpha_n)|^2 \, d\alpha_n \right\}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_m(\psi)|^2 \, d\psi \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_n(\psi)|^2 \, d\psi = mn.$$

Hence $$S_2 \leqslant \sum_{m \neq n} (mn)^{1-2\beta} \leqslant \left(\sum m^{-\frac{1}{2}}\right)^2 = A. \tag{27.7}$$

Finally, it follows from (27.5), (27.6), and (27.7) that

$$\mathfrak{A}_\alpha\{\mathfrak{M}_4^4(f)\} \leqslant A \leqslant A^4,$$

which is (27.2).

We have thus proved what we stated in § 24, though we have not produced a definite f and g. It will be observed that we have made no use of the Littlewood-Paley theorems in this part of our work.

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